Actuarial Society of India

EXAMINATIONS

June 2005

ST6 – Finance and Investment B

Indicative Solution

1. a) We can replicate the derivative payoffs using an initial portfolio consisting of f shares and y units of cash. [½]

To replicate correctly, we need:

$$fS_0u + ye^{rdt} = h_u$$

and $$fS_0d + ye^{rdt} = h_d$$ [1]

Subtracting these and then substituting back gives:

$$fS_0(u-d) = h_u - h_d \Rightarrow f = \frac{h_u - h_d}{S_0(u-d)} \Rightarrow y = \frac{h_d u - h_u d}{e^{rdt}(u-d)}$$ [½]

b) Formula for V_0

If we assume that the markets are arbitrage-free, the value of the replicating portfolio must equal the initial value of the derivative.

So: $V_0 = fS_0 + y$ [½]

substituting the values derived above:

$$V_0 = \frac{h_u - h_d}{S_0(u-d)} S_0 + \frac{h_d u - h_u d}{e^{rdt}(u-d)}$$

$$= \frac{e^{rdt}(h_u - h_d) + (h_d u - h_u d)}{e^{rdt}(u-d)}$$

$$= \frac{(e^{rdt} - d)h_u - (e^{rdt} - u)h_d}{e^{rdt}(u-d)}$$

$$= \frac{\frac{(e^{rdt} - d)}{(u-d)} h_u + \frac{(u - e^{rdt})}{(u-d)} h_d}{e^{rdt}}$$ [2]

If we let $q = \dfrac{e^{rdt} - d}{u - d}$, so that 1-q = $\dfrac{u - e^{rdt}}{u - d}$, this simplifies to:

$$V_0 = \frac{qh_u + (1-q)h_d}{e^{rdt}}$$ [½]

c) Relationship to derivative pricing formula

The formula derived in part (b) corresponds directly to the derivative pricing formula. [½]

H is the payoff, which is either h_u or h_d [½]

Q is the risk-neutral probability measure, ie the probability q and 1-q. [½]

T-t is the remaining life of the derivative, which equals *dt*. [½]

F_t is the filtration, which represents the history of the share price up to time t. [½]

However, at time 0 there is no history. [½]

d. Explain why d< e^r < u

An up-step must result in a higher price than a down step. So we must have d < u. [½]

If e^r<d(<u) then the share would always produce a higher return than an investment in cash. We could therefore create an arbitrage by borrowing cash to buy shares. [½]

If (d<)u<e^r then the share would always produce a lower return than an investment in cash. We could therefore create an arbitrage by short selling shares and investing the proceeds in cash. [½]

This inequality is equivalent to the requirement that the risk neutral probability q satisfies 0 < *q* < 1. [½]

[total 10]

2 a) Mean = 20%

¾ u + ¼ d = 1.2

d = 4.8 – 3u ------------- [½]

variance = 15%

var (x) = $E(x^2) – [E(x)]^2$

$¾ u^2 + ¼ d^2 – 1.2^2 = 0.15^2$

$3u^2 + d^2 = 5.85$ ----------------- [½]

From (1) and (2)

$3u^2 + (4.8 – 3u)^2 = 5.85$

$12u^2 – 28.8u + 17.19 = 0$

$$u = \frac{\sqrt{28.8^2 - 4x12x17.19}}{24} = \frac{28.8 \pm \sqrt{4.32}}{24}$$ [½]

u = 1.2866 and d = 0.9402 ----------------- [½]

u = 1.1134 and d = 1.4598 ------------------- [½]

since u > d, we have

u = 1.2866 and d = 0.9402 ---------------------- [½]

[Total 3]

b) (i) value of the call option

$$q = \frac{e^{0.06} - 0.9402}{1.2866 - 0.9402} = \frac{1.0618 - 0.9402}{0.3464} = \frac{0.1216}{0.3464} = 0.3510$$

h_u = Max (1.2866 x 300 – 325, 0)

h_u = Max (60.98, 0) = 60.98

h_d = Max (0.9402 x 300 – 325, 0) = 0

$$c = \frac{qh_u + (1-q)h_d}{e^{rdt}} = \frac{0.3510x60.98 + (1-0.3510)x0}{e^{0.06}}$$ ---------- [2½]

$$c = \frac{21.40}{1.0618} = 20.16$$

ii) value of put option

h_u = Max (325 – 1.2866 x 300, 0) = 0

h_d = Max (325 – 0.9402 x 300, 0)

h_d = Max (42.94, 0) = 42.94

$$p = \frac{qh_u + (1-\ell)h_d}{e^{rdt}} = \frac{0.3510x0 + (1-0.3510)x42.94}{e^{0.06}}$$

$$p = \frac{27.8681}{1.0618} = 26.25$$ ---------------- [2½]

c) put – call parity relationship:

$p + S_0 = c + Ke^{-\text{rät}}$

L.H.S = 26.25 + 300 = 326.25

R.H.S = 20.16 + 325 x $e^{-0.06}$

= 20.16 + 306.09

= 326.25

LHS = RHS ------------------- [2]

[Total 10]

3. a) (i) The form of the Black-Scholes partial differential equation is:

$$\frac{\partial f}{\partial t} + (r-q)S_t \frac{\partial f}{\partial S_t} + 1/2s^{2} \frac{\partial^2 f}{\partial S_t^2} = rf$$

The original model described by Black and Scholes made no allowance for dividends. The PDE for this mode (with q = 0) is therefore:

$$\frac{\partial f}{\partial t} + rS_t \frac{\partial f}{\partial S_t} + 1/2s^{2} \frac{\partial^2 f}{\partial S_t^2} = rf \qquad [½]$$

Or, in terms of the "Greeks":

$\Theta + rS_t\text{Ä} + ½\, \sigma^2 \Gamma = rf$

Any of these forms would have been acceptable in the exam.

The variables are defined as follows:

- t represents the current time [½]
- f is the derivative price at time t [½]
- S_t is the price of the underlying security at time t [½]
- r is the risk-free rate of interest (expressed in continuously –compounded form) [½]
- σ is the volatility of the price of the underlying security, ie the standard deviation of the log returns [½]

[Total 3]

ii) The main assumptions of this model are:

- The security price conforms to geometric motion (the lognormal model).
- In particular, this assumes that the volatility is constant.
- The primary securities market, the derivatives market and the bond market are all arbitrage-free.
- The markets are complete, ie it is possible to replicate the payoff of the derivative using appropriate amounts of the basic security and cash.
- The markets are frictionless, ie there are no dealing costs and the effect of taxes can be ignored.
- It is possible to deal instantaneously in the shares in any amounts (however large or small) and to hold positive or negative positions in the shares and/or in cash.

[6 x ½ =3]

iii) The parameter σ represents the volatility of the underlying security over the life of the derivative. [½]

Unlike the other parameters, this is not directly observable. So it requires an assumption. [½]

The calculated derivative price is very sensitive to the assumed volatility, so it is important to use reliable figure. [½]

Two methods are commonly used to select an appropriate value of σ:

- Calculate a historical estimate based on the observed past prices of the security over a period of time of similar length to the life of the derivative being priced. [½]

- Calculate an implied volatility from the current market price of another derivative on the same security. The implied volatility is the price that reproduces the observed marked price. [1]

[Total 3]

b) Ito's lemma is applicable if the stock price S_t follows a diffusion process, ie if it obeys a stochastic differential equation of the form:

$$dS_t = m(S_t, t)dt + s(S_t, t)dB_t$$

where B_t represents standard Brownian motion. [1]

In particular, the values of the functions $\mu(\ )$ and $\sigma(\ )$ must not depend on any other variables apart from the current stock price and the current time (ie all other quantities such as the risk-free rate are assumed to be constant). [1]

There are technical conditions limiting the "size" of $\mu(S_t, t)$ and $\sigma(S_t, t)$ to ensure that the corresponding integrals coverage. [½]

If Ito's lemma is used to find the SDE of a function $f(S_t, t)$, then f must be twice-differentiable with respect to St and once differentiable with respect to t. [½]

[3]

[Total 12]

4. a) Value of a floating-rate par bond

By definition, a floating-rate bond will pay the current market rate of interest on the principal amount. (We're assuming here that the floating rate is actually equal to the current market rate, not for example 1% above.) [½]

So, unlike with a fixed interest bond, there is no discrepancy between the market price and the redemption amount (which, for a par bond, equals the nominal amount). [½]

The only reason the value may differ from the nominal amount is because of interest that has accrued since the last payment date.

[½]

As an analogy, if someone takes out a variable-rate interest mortgage, just after any interest payment has been made, the amount of loan outstanding (=the value) will equal exactly the amount originally borrowed. [½]

A par bond is a bond whose current price is equal to its redemption amount.

[Total 2]

b) i) "Tenor"

The interest rate that determines the cashflows paid under an interest rate cap or floor is reset periodically to reflect the current market interest rate. [1]

The tenor is the length of time between resets, typically three months. [1]

ii) "Caplets" and "floorlets"

With an interest rate cap, a payment is made on each interest payment date if the benchmark interest rate exceeds the cap rate at the start of that of that period. The payoff has the same form as a call option on the interest rate, exercisable on that date – but with the pay off payable at the next reset date. Each such payment is called a "caplet". [1]

With an interest rate floor, payments are made if the benchmark rate is below the floor. The payments under an interest rate floor have the same payoff as a put option, and are called "floorlets". [1]

[4]

[Total 6]

6. a) i. Credit risks with an option

The writer of the option has no credit exposure to the purchaser because the option premium is paid up-front. [1]

The purchaser has a credit exposure because the writer might default on the profit payment payable on exercise. [1]

ii. Credit risks with a future

Both parties have a potential credit exposure to the other because the price of he underlying may move up or down in the future. [1]

At any point in time, one of the parties will have a current credit exposure to the other, depending on whether the cash price is above or below the forward price. [1]

[total 4]

b) i. The easiest way is to ensure that contracts are only made with counterparties who have high credit ratings. [1]

ii. The easiest way to reduce the credit exposure would be to require daily marking to market using a margin system. [1]

[Total 2]

c) Under this arrangement, XYZ bank is holding the bond as collateral for the cash it has lent. [1]

The bank is exposed to credit risk because the pension fund might not buy the bond back, as agreed. [1]

It would then also be exposed to credit risk if the bond does not turn out to have the value expected. [1]

The risks could be reduced by:

- Insisting that the zero coupon bond is one that is backed by the government [1]

- Marking to market [1]

[5]

[total 11]

7. a. Delta of call = $\Phi(d_1)$

Delta of put = $\Phi(-d_1)$

$$d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{s^2}{2}\right)T}{s\sqrt{T}}$$

$$d_1 = \frac{\ln\left(\frac{30}{30}\right) + \left(0.05 + \frac{0.0625}{2}\right)x1}{0.25\sqrt{1}}$$

$d_1 = 0.3250$
$\Phi(0.325) = 0.6274$
$\Phi(-0.325) = 0.3726$
Delta of call = 0.6274
Delta of put = -0.3726 [1]

b) If the investor holds 200 shares, the delta of his portfolio is 200
Let x number of put option are required to hedge these shares
200 – 0.3726x = 0

$$x = \frac{1883}{0.3726} = 536.77 = 534(say)$$

He needs to buy the puts, so that overall delta of his portfolio is:
200 x 1 + 536.77 (-0.3726) 0 [2]

c) Let y number of call options are required to hedge 300 shares:
300 – 0.6274y = 0; $y = \frac{300}{0.6274} = 478.16 \approx 478$ [2]
She needs to sell the call

[Total 5]

8. a) The oil consumer has a requirement of 80,000 barrels and he has hedged his position with 20 long futures contacts (20 x 2000 = 40,000 barrels)

$h = \frac{40000}{80000} = 0.5$ -------------------------- [1]

The oil speculator has purchased 40000 barrels and has hedged his position with 30 futures contracts (30 x 2000 = 60000)

$h = \frac{60000}{40000} = 1.5$ ---------------------------- [1]

b) Minimum variance edge ratio

$h_{\min} = r\frac{s_S}{s_f} = 0.78x\frac{2}{3} = 0.52$ ------------------------- [2]

c) Comment

The oil consumer is quite close to the minimum variance hedge ratio. This position would minimize the risk associated with an unexpected increase in the price of oil. [1]

If the price of oil increases, the consumer will have to pay more to meet the requirement for the 80,000 barrels, but this will be largely offset by the gain in the value of the 20 futures contracts. [½]

The oil speculator's position is a long way from the minimum variance hedge ratio. This position exposes the speculator to large losses if there is an unexpected increase in the price of oil. [1]

If the price of oil falls, the value of the speculator's existing holding of 40,000 will fall, but this will be more than outweighted by an increase in the value of the futures contracts.

9. a) When ô = 0, we have f (0) = R
R denotes the short rate, we have R = 5%
As t ∞, we have f (∞) = L
L denotes the long rate, we have L = 6% ------------------- [2]

b) For a stationary point, we have $f^{/}$ (ô) = 0

$$f^{/}(t) = \frac{d}{dt}\left[\mathrm{R}e^{-at} + L\left(1-e^{-at}\right) + \frac{b}{a}\left[e^{-at} - e^{-2at}\right]\right] = 0$$

(L – R) α + β [2e$^{-áô}$ – 1] = 0

$$b = \frac{(R-L)a}{2e^{-at} - 1}$$

when ô = 15

$$b = \frac{(R-L)a}{2e^{-15a} - 1}$$

---------------------------------- [3]

c) Since $R<L$ and α > 0, we have (R-L) α < 0,
Thus, for β to be positive,
$2\,e^{-15\alpha} - 1 < 0$
$2\,e^{-15\alpha} < 1$
$e^{-15\alpha} < ½$
$e^{-15\alpha} < 0.5$
$\alpha > 0.0462$
If $\alpha > 0.0462$, $\beta > 0$

$\alpha < 0.0462$, $\beta < 0$ ---------------------- [2]

d) Limitations of the vasicek model

- If α is above a certain critical value, then β is positive and the curve is humped, ie it has a maximum value.
- This humped shape of yield curve is commonly observed.
- If α is below the critical value, then β is negative and the stationary point is a minimum.
- This shape of yield curve would not normally be observed, because it is inconsistent with the principle of no-arbitrage.
- There is a certain amount of flexibility in terms of shapes of curve that can be obtained with this model.

[5]

10. a) Let S_2 be the value NSE Nifty at the end of 2 years. The building society must pay out on 2 bonds.

$$H = \begin{cases} 1.4x2000 = 2800, S_2 \geq 2800 \\ S_2, 20x < S_2 < 2800 \\ 20x, S_2 \leq 20x \end{cases}$$

x 1.4 [1]

If an investor buys the bond then MHS can invest the money in Nifty so that they are not exposed to movement of Nifty. MHS can hedge its position by:

a) buying a put option with an exercise price of 20x and time to maturity of 2 years. The put option will cost p
b) by writing a call option with an exercise price of 2800 and time to maturity of 2 years. The call option is priced at c.

Paaff from the portfolio is [1]

	S_2 20x	$20x < S_2 < 2800$	S_2 2800
Nifty	S_2	S_2	S_2
Put Purchased	$20x - S_2$	0	0
Call written	0	0	$-(S_2 - 2800)$
	20x	S_2	2800

If c p, MHS will not make a loss.
If c = p, MHS will neither make a profit nor loss. Thus, first find out the price of c and then find out the value of x such that c = p [1]

b) Using Black–Scholes model, we have

$$d_1 = \frac{\ln(So/K) + \left(r + \frac{s^2}{2}\right)2}{s\sqrt{2}}$$

$$d_1 = \frac{\ln\left(\frac{2000}{2800}\right) + \left(0.06 + \frac{0.04}{2}\right)x2}{0.20\sqrt{2}}$$

$$= \frac{-0.3365 + 0.16}{0.2828} = -0.6240$$

$$d_2 = d_1 - s\sqrt{2}$$

= - 0.6241 – 0.2828 = -0.9068

$\Phi(-0.6240) = 0.2663$

$\Phi = (-0.9068) = 0.1823$

$C = S_0\, \Phi(d_1) - Ke^{-2r}\, \Phi(d_2)$

$= 2000 \text{ x } 0.2663 - 2800e^{-2x0.06} \text{ x } 0.1823$

= 532.60 – 452.72 = 79.88 [1]

We need to find the value of x such that p = 79.88

Try K = 1900

$$d_1 = \frac{\ln\left(\frac{2000}{1900}\right) + \left(0.06 + \frac{0.04}{2}\right)x2}{0.20\sqrt{2}} = 0.7471$$

$d_2 = d_1 - ó \quad 2 = 0.7471 - 0.2828 = 0.4643$ [1]

$P = k\ \Phi\ (-d_2)\ e^{-2r} - S_0\ \Phi(-d_1)$

P = 1900 x 0.8869 x 0.3213 – 2000 x 0.2275

P = 541.43 – 455 = 86.43

This is higher than 79.88, so we have a lower value of K = 1800 [1]

$$d_1 = \frac{\ln\left(\frac{2000}{1800}\right) + \left(0.06 + \frac{0.04}{2}\right)x2}{0.20\sqrt{2}}$$

$$= \frac{0.1054 + 0.16}{0.2828} = 0.9385$$

$d_2 = d_1 - s\sqrt{2} = 0.6557$

$p = 1800\ e^{-2x0.06}\ \Phi(-0.6557) - 2000\ x\ \Phi\ (-0.9385)$

= 1596.46 x 0.2560 – 2000 x 0.1740

= 408.69 – 348 = 60.69

we have to find value of k for p = 79.88. Using linear interpolation, we get

$$K = 1800 + (1900 - 1800)\left(\frac{79.88 - 60.69}{86.43 - 60.69}\right)$$

$$= 1800 + 100x\frac{19.19}{25.74}$$

= 1800 + 74.55

= 1874.55 [1]

$$x = \frac{1874.55}{2000}x100 = 93.73$$ [1]

So MHS can set a guarantee level at 94% and use the hedging portfolio described to avoid making a loss.

c) Lower K = 0.90 x 2000 = 1800

p (K = 1800) = 60.69

c (K = 2800) = 79.88

C > P

Profit = C – P

= 79.88 – 60.69 = 19.19 [1]

NSE Nifty = Value of two bonds
Expected profit on two bonds = 19.19
Expected profit per bond = 19.19/2 = 9.60 [1]

d. i). The building society is using a static hedging policy that is only guaranteed to work if it maintains this portfolio until the end of the three years. [1]

If it has to unwind its position before then, it stands to make a loss. [½]

Imposing this restrictions avoids this problem. [½]

ii) The building society could offer to pay a "surrender value" on early redemption based on the current value of the hedged portfolio, but this would not provide consumers with the guarantee they would expect from a policy of this type. [½]

Since the short holding in the call option could potentially make an unlimited loss, the value of the hedging portfolio could actually be negative! [½]

There could be a selection effect if the stock market made a very large unexpected movement and a large number of bondholders elected to redeem early at what would be an unfavourable time for the building society. [½]

Alternatively, building society could set aside some of the bondholders' initial cash and invest the remainder in a hedged portfolio. However, this would mean that they would have to reduce the value of *x* in the guarantee, making the bond less attractive to bondholders. Also, the guaranteed surrender value would be minimal. [½]

e) Other sources of risk

- In reality, options with the exact strike prices are unlikely to be available. So the hedge will not be perfect because of basis risk.

- Similarly, it may not be possible to find a government bond with the precise period required for investing the cash. Using another type of investment may introduce a credit risk.
- If traded options are used, the credit risk on these should be negligible. However, if OTC options are used to match the strike price and timing more precisely, there may be a risk that the counterparty will default.
- Operational risk is always present when launching a new product.
- Credit risk exposure to the investors is avoidable, provided that bond certificates are not issued until investors' funds have cleared.

[Any 4, 4 x ½ = 2]

f) According to our assumptions, any expected future growth in the shares included in the index should already be reflected in the prices at which these are trading, and hence in the value of the index. So the rate of growth is taken into account implicitly. [1]

Another way of looking at this is to say that, since the markets can be assumed – to a good approximation – to be arbitrage-free, there is a probability measure under which the discounted value of the index is a martingale. This eliminates the effect of any drift (ie growth) from the calculations. [1]

[Total 18]

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