June 2005
CT8 - Financial Economics

Q.1 a)

1) The expected return on the bond is given by
$90 * 8\% + 0.05 * 4\% + 0.05*0\% = 7.4\%$ **[1]**
So the downside semi-variance is
$(7.4 - 4)^2 * 0.05 + (7.4 - 0)^2 * 0.05 = 3.316\%$ **[1]**

2) The probability of receiving less than 5% is equal to the sum of probabilities of receiving 4% and 0% i.e. 0.10. **[1]**

3) Expected conditional shortfall
The expected shortfall below the risk-free rate of 5% is given by
$(5 - 4) * 0.05 + (5 - 0) * 0.05 = 0.30\%$ **[1]**
Hence the expected shortfall below risk free or conditional on a shortfall occurring is

$$\frac{\text{expected shortfall}}{\text{shortfall probability}} = \frac{0.30\%}{0.10} = 3\%$$ **[2]**

We can also observe this directly by noting that, given that there is a shortfall it is equally likely to be 1% or 5% so the expected conditional shortfall is 3%.

Total [6]

b) Usefulness of down-side semi-variance as a measure of investment risk.

- It gives more weight to downside risk ie. variability of investment returns below the mean, than upside risk **[0.5]**
- In fact it completely ignores risk above the mean **[0.5]**
- This is consistent with the investor being risk-central above the mean which may not be the case in practice. **[0.5]**
- The mean is an arbitrary bench which might not be appropriate for a particular investor. **[0.5]**
- If investment returns are symmetrically distributed about the mean (as they would be, for eg., with a normal distribution) then it will give equivalent results to the variance based analysis. **[0.5]**
- It is much less mathematically tractable than the variance. **[0.5]**

(any four) **Total [2]**
G. Total [8]

Q.2

a) In the context of CAPM, the market price of risk is defined as

$$\frac{E_M - R_F}{\sigma_M}$$ where **[1]**

E_M = expected return on the market portfolio

R_F = risk-free rate of return

σ_M = standard deviation of market portfolio returns

It is additional expected return that the market requires to accept an additional unit of risk as measured by the standard deviation of the return. **[1]**

It is equal to the gradient of the capital market line in the E - σ space. **[1]**

Total [3]

b) Investors will consider the mean of variance of return only:

When they are not concerned with either their liabilities (or do not have any specific liabilities) or marketability and either. **[2]**

Their utility function is quadratic (so that the maximization expected utility is equivalent to maximizing function that depend only upon the mean and variance of expected wealth) Or **[1]**

Asset returns are known to be normally distributed (so that the distribution of returns is fully specified by the mean and variance) **[1]**

Total [3]

CAPM as a single-index model

Consider the single index model in which the single index is the return on the market portfolio. Then

$R_i = a_i + b_{i,1}R_M + C_i$

Thus

$\text{Cov}(R_i, R_M) = Cov\left[a_i + b_{i,1}R_M + C_i, R_M\right]$

$= b_{i,1}\sigma_M^2$ **[0.5]**

Hence $\beta_{\text{i}} = \dfrac{\text{Cov}(R_i, R_M)}{\sigma_M^2} = b_{i,1}$ **[0.5]**

Applying the same argument that is used to derive the arbitrage pricing theory result, we can show that

$E_i = \lambda_0 + \lambda_1 b_{i,1}$ **[0.5]**

Now for the risk, free asset, β_{i}=0 and so

$E_i = \lambda_0 + \lambda_1\beta_{i,1} = \lambda_0 + \lambda_1 * 0$

$\lambda_0 = r$ **[0.5]**

Thus $\lambda_0 = r$ Additionally for the market portfolio

$E_M = \lambda_0 + \lambda_1\beta_{i,1} = \lambda_0 + \lambda_1 * 1 = \lambda_0 + \lambda_1$

So $\lambda_1 = E_M - \lambda_0 = (E_M - r)$ **[0.5]**

Hence
$$\begin{aligned} E_i &= \lambda_0 + \lambda_1 b_{i,1} \\ &= r + \beta_i(E_M - r) \end{aligned}$$ **[0.5]**

Which is the security Market Line relationship in Capital Asset Pricing Model.

Thus the Capital Asset Pricing Model and Arbitrage Pricing Theory are consistent if

- Returns are generated by a single –index model in which the return on the market portfolio is the index; **[0.5]**
- There exists a risk-free rate of return. **[0.5]**

Total [4]

G. Total [10]

Q.3 a)

i) The lognormal model of security prices is consistent with weak form market efficiency. **[0.5]**

ii) This is because log returns over non-overlapping time intervals are assured to be independent in the model. **[0.5]**

iii) Thus, knowing the past patterns of returns cannot help you predict future returns. **[0.5]**

iv) In contrast the Wilkie model is not consistent with weak form model efficiency. **[0.5]**

v) This can be shown by using the model to project the equity risk premium – the excess expected total return on equities compared to index-linked govt. bonds. **[0.5]**

vi) If markets were efficient we would expect the risk premium projected by Wilkie model to remain in a narrow range. Otherwise, excess profits could be earned by holding equities when the risk premium is high and holding index-linked govt. bonds otherwise. **[1]**

vii) The wide variation in the equity risk premium projections generated by the Wilkie model is equivalent to ranging that the equity and index-linked govt. bond markets are not efficiently priced relative to each other. **[1]**

Total [5]

b) The integral can be thought of as the (limiting) sum of the small elements tdB_t .

The $dB'_t r$ are random with mean '0'and variance dt. So the expected value of each element is zero and the variance is t^2dt

So E(I) = 0 **[1]**

And $\text{Var(I)} = \int_0^1 t^2 dt = \frac{1}{3}$ **[2]**

Total [3]

G Total [8]

Q.4 a)

i) Absolute risk over main

$$A(w) = \frac{U''(w)}{U'(w)}$$ **[0.5]**

ii) Relative risk over main

$$R(w) = -w\frac{U''(w)}{U'(w)}$$ **[0.5]**

Total [1]

b)

i) Assumptions underlying Mean-Variance part theory
Investors select their portfolios on the basis of expected return and variance of return.(or the returns on assets are distributed with a multive normal distribution)

ii) Investors are risk-averse

iii) Investors are never satisfied

iv) Investors make their investment decisions over a single time horizon.

Total [2]

c)

$$X_j + X_s = 1$$

$$\sigma^2 = X_j^2\sigma_j^2 + \left(1 - X_j\right)^2 \sigma_s^2 + 2X_j\left(1 - X_j\right)\rho\sigma_j\sigma_s$$

$$\frac{\partial\sigma^2}{\partial X_j} = 0 \quad \text{for Stationary point}$$

$$\Rightarrow 2X_j\sigma^2 - 2\left(1 - X_j\right)\sigma_s^2 + 2\rho\sigma_j\sigma_s - 4X_j\rho\sigma_j\sigma_s = 0$$

$$\Rightarrow X_j = \frac{\sigma_s^2 - \rho\sigma_j\sigma_s}{\sigma_j^2 + \sigma_s^2 - 2\rho\sigma_j\sigma_s}$$

$$= \frac{(25)^2 - (0.5)(22)(25)}{22^2 + 25^2 - 2(0.5)(22)(25)}$$

$$= \frac{625 - 275}{484 + 625 - 550}$$

$$= \frac{350}{559}$$

$$= 0.6261$$

$$= 62.61\% \text{ in j}$$

Hence for j = 62.61%

For s = 37.39% **[4]**

$$r_p = (62.61)(14\%) + 37.39(16\%)$$

$$= 0.147476$$

$$= 14.75\%$$ **[1]**

$$\sigma_p = \Big[\left(0.6261\right)^2(0.22)^2 + (0.3739)^2(0.25)^2$$

$$+ 2(0.6261)(0.3739)(0.22)(0.25)(0.5)\Big]^{\frac{1}{2}}$$

=20.1459% = S.D.

Hence σ^2 = 405.8587 **[3]**

Total [8]

G Total [11]

Q.5

a)

i) Probability that B_2 takes a positive value . B_2 is likely to be positive or negative (and has zero probability of being exactly zero)

So $P(B_2 > 0) = \frac{1}{2}$ **[1]**

ii) B_2 has N(0,2) distribution so prob($-1 < B_2 < 1$) is

$$P(-1 < B_2 < 1) = \Phi\left[\frac{1}{\sqrt{2}}\right] - \Phi\left[\frac{1}{\sqrt{2}}\right]$$

= 0.760 – (1 – 0.760)

= 0.520 **[2]**

iii) Probability that B_1 and B_2 both take positive values is

We can write the required probability as

$p = P(B_1 > 0, B_2 > 0) = P[B_1 - B_0 > 0,\ B_2 - B_1 > -B_1]$

If we now write $x = B_1 - B_0$ and $y = B_2 - B_1$ then we know from the properties of Brownian motion that x and y are independent, each with N(0,1) distribution.

So the required probability is

$P = P(x > 0, y > -x)$

Since the range of values of y depends on the value of x, we must use a double integral to evaluate this **[4]**

$$p = \int_0^{\alpha} dx \Phi(x) \int_{-x}^{\alpha} dy \Phi(y)$$

Integral over y is $\left[\Phi(y)\right]_{-x}^{\alpha}$

$= 1 - \Phi(-x)$

$= \Phi(x)$

So $p = \int_0^{\alpha} dx \Phi(x)\ \Phi(x)$

$$= \left[\frac{1}{2}\left[\Phi(x)\right]^2\right]_0^{\alpha}$$

$$= \frac{1}{2}\left[1^2 - \left(\frac{1}{2}\right)^2\right]$$

$$= \frac{1}{2}\left[1 - \frac{1}{4}\right]$$

$$= \frac{1}{2}\left[\frac{3}{4}\right]$$

$$= \frac{3}{8}$$

iv) $B_{100} - B_{50} \sim N(0, 50)$
Since $B_{50} = 3.04$ B_{100} will be negative if
$B_{100} - B_{50} < -3.04$
This has probability of

$$\phi\left(-\frac{2.04}{\sqrt{50}}\right) = \phi\left(-0.43\right) \simeq \frac{1}{3}$$

[1]

[Total 1+2+4+1 =8]

b) Recall that if the price of a security S_t follows a geometric Brownian motion then

$$dS_t = \alpha S_t dt + \sigma S_t dB_t$$

Where α and σ are constants and B_t is a Standard Brownian motion. **[1]**

Recall that Ito's lemma states that if F_t is a twice continuously differentiable function of S_t and

$$dS_t = \alpha(S_t, t) + b(S_t, t) dB_t$$

then $dF_t = \frac{\partial F}{\partial t} dt + \frac{\partial F}{\partial S} dS_t + \frac{1}{2}\frac{\partial^2 F}{\partial S^2} dS_t^2$

So if F = log (S), then

$$\frac{\partial F}{\partial t} = 0, \quad \frac{\partial F}{\partial S} = \frac{1}{S}, \quad \frac{\partial^2 F}{\partial S^2} = \frac{1}{S^2}$$ **[1]**

If S_t follows a geometric Brownian motion, then substituting these **[2]** into the previous equation gives

$$dF_t = 0dt + \frac{1}{S_t}\left[dS_t dt + \sigma S_t dB_t\right] - \frac{1}{2S_t^2}\left[\alpha S_t dt + \sigma S_t dB_t^2\right]$$

$$= \alpha dt + \sigma dB_t - \frac{1}{2S_t^2}\left(\sigma^2 S_t^2 dt\right)$$

$$= \left[\alpha - \frac{\sigma^2}{2}\right]dt + \sigma dB_t$$

If we set h = $\alpha - \frac{\sigma^2}{2}$, then

$dF_t = hdt + \sigma dB_t$

Integrating over the time interval (t, u) and remembering that the increments of standard Brownian motion are stationary, independent and Gan   finally gives

$$\log(S) - \log(S_t) \sim N\left[h(0-t), \sigma^2(u-t)\right]$$

which is the standard formula for the continuous-time lognormal model of security prices. **[1]**

Total [1+1+2+1 = 5]

Total [13]

Q.6 a) A market is said to be arbitrage free if there are no opportunities for making a risk-free project by taking advantage of inconsistencies in the prices of different aspects. **[1]**

b) i) Values of v and d

Equating the mean and variance of the returns gains:

$$\frac{2}{3}v + \frac{1}{3}d = 1.1 \Rightarrow d = 3.3 - 2v$$

and $\frac{2}{3}v^2 + \frac{1}{3}d^2 - 1.1^2 = 0.1^2 \Rightarrow 2v^2 + d^2 = 3.66$ **[1]**

Eliminating d from there simultaneous equation gives

$$2v^2 + (3.3 - 2v)^2 = 3.66$$

i.e. $\text{b}v^2 - 13.2v + 7.23 = 0$ **[1]**

Solving this using quadratic equation gives

$$u = \frac{13.2 \pm \sqrt{13.2^2 - 4(6)(7.23)}}{2(6)}$$

$$= \frac{13.2 \pm \sqrt{0.72}}{12}$$

So v = 1.17071 & d = 0.95858 **[1]**

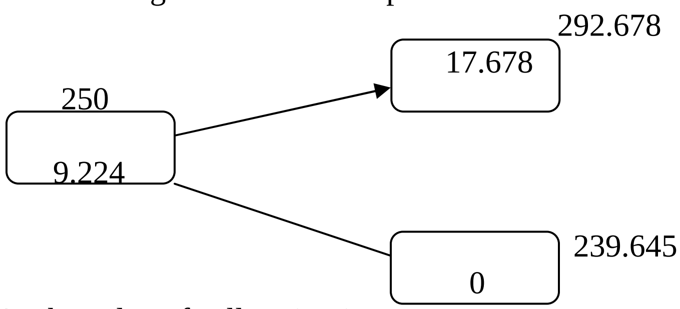
Or u = 1.02929 & d = 1.24142 **[1]**

Since u > d we can eliminate the record pair of values and canclude
u = 1.17071 & d = 0.95858 **[1]**

ii) The risk-neutral probability of an up-movement is

$$q = \frac{e^{0.075} - 0.95858}{1.17071 - 0.95858} = 0.56241$$ **[1]**

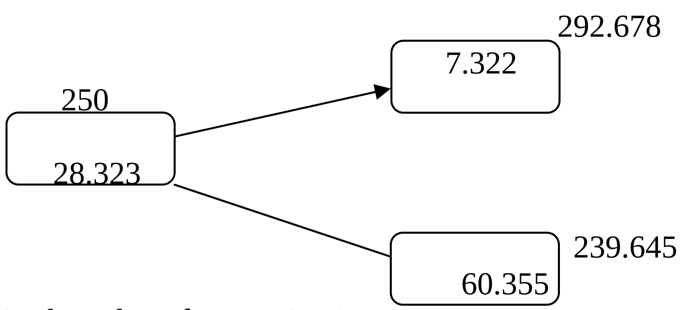
The tree diagram for the call option looks like this



So the value of call option is

$$\left(0.56241*17.678 + 0\right)e^{-0.075} = 9.224$$ **[2.5]**

The tree diagram for the call option looks like this



So the value of put option is

$$\left(0.56241*17.678 + 0.43759*60.355\right)e^{-0.075} = 28.323$$ **[2.5]**

Total [6]

Q.7 a) Volatility refers to the "random" variations observed in the nearest price of the share. **[1]**

b) This means that S_t the share price at time t, can be considered to be a random variable that obeys the Stochastic differential equation

$$dS_t = S_t(hdt + \sigma dZ_t)$$

where Z_t represents a standard Brownian motion.

Another way of expressing it, is to say that the distribution of

$\log = \frac{S_t}{S_0}$ is $N\left(h_t, \sigma_t^2\right)$ and that movement in S_t is non-overlapping time intervals are statistically independent.

[3 for either formulation]

This may not be realistic in practice for any underlying asset that experiences sudden changes (discontinuous jumps) in price or whose drift or volatility is not constant. **[1]**

Total [4]

c)

i) Value the put option using hedging argument

The option will be worth \$ 0.50 if the share price falls to \$ 1.50. It will be worthless if the share price rises to \$ 2.50.

A purchaser of the option can create a hedged position by buying n shares where:

$$50 + 150 * n = 0.250 * n$$

$$\Rightarrow n = \frac{1}{2}$$

With this value of n, both sides of this equation equal 125 i.e. the portfolio would be worth \$ 1.25, whichever way the share price moves. So a straight forward equation of value now tells us that the price of the put option 'p' satisfies the equation

$$p + 219 * \frac{1}{2} = 125e^{-0.01} \Rightarrow p = 14.26$$

[4]

ii) Value the put option using risk-neutral valuation

The risk-neutral probability q of an upward movement in the share price can be determined from

$$q * 250 + (1 - q) * 150 = 221e^{0.01}$$

$$\Rightarrow q = e^{-0.01} = \frac{221e^{0.01} - 150}{250 - 150} = 0.73221$$

So the value of the option is

$$p' = e^{-0.01}\left[q * 0 + (1 - q) * 50\right] = 13.26$$

[2]

iii) Estimate the delta using the results from parts (i) and (ii) the option's delta is therefore approximately:

$$\Delta = \frac{\partial f}{\partial S} \simeq \frac{13.26 - 14.26}{221 - 219} = -\frac{1}{2}$$

This is the value we would expect for delta since creating a deta-hedged position, as we did in part(i) where we found that n = $\frac{1}{2}$ involves purchasing or selling Δ shares. **[1]**

Total [7]

Q.8 a) A one factor model is one in which interest rates are assumed to be influenced by a single source of randomness. **[1]**

The prices of all bonds (of all maturities) and interest rate derivaties must therefore move together. **[0.5]**

The randomness is usually modeled as an Ito process. **[0.5]**

The stochastic differential equation for r(t) has the following form under the real would probability measure P:

dr(t) = a (t , r(t)) dt + b(t , r(t)) dw(t)

where a( ) and b( ) are approximately chosen function. **[1]**

Limitations of one-factor models:

The prices of bonds of different terms in the real world are not observed to be perfect by correlated, always moving together. **[1]**

Sustained periods have been observed historically with all combinations of high/low interest rates and high/low volatility , this would appears to be inconsistent with a one-factor model. **[1]**

Some interest rate products are explicitly dependent on other variables which would be expected to introduce separate source of randomness. For example, some interest rate derivatives are based on both UK and US interest rates which are not perfectly correlated. **[1]**

Total [6]

b) Advantages / disadvantages of the redington model:

The advantages of this model are :

- It is mathematically tractable(easy to do calculations) **[1]**
- It provides a sufficiently accurate approximation in situations when the term of the investments is not crucial, e.g. for long term investors such as pension funds. **[1]**

The disadvantages of this model are :

- It is not arbitrage –free
- It does not accurately represent real life yield curves, which are not flat. **[1]**

Total [4]

c)

i) Expression for B_t

B_t is the accumulated value at the time t of an initial investment of 1 unit of cash.

So $B_t = e^{rt}$ **[1]**

ii) To show that the discounted cash process is a martingale

The discounted cash process is therefore

$e^{-rt}B_t = e^{-rt}e^{rt} = 1$ **[2]**

which trivially satisfies the martingale equation

$E_Q\left[e^{-rT}B_T \mid F_t\right] = e^{-rt}B_t$

iii) Deduce that any self-financing portfolio is also a martingale. We have established that the discounted values of the components (the shares and the cash) of such a portfolio are martingales .
So any multiple of these will also be a martingale. **[1.5]**
Also if we "rebalance" the portfolio by making switches from cash to shares or vice versa, this will not affect the martingale property provided that we don't put any money in or take any out. **[1.5]**
A good intuitive may to think of martingales here is to that, on average, they don't drift up or down. So, if our(discounted) cash and shares are not drifting up or down neither will any combination of them.

Total [6]
G Total [16]

Q.9 a) In a recombining binomial tree u and d, the proportionate increase and decrease in the underlying security price at each step, are assumed constant throughout the tree. As a result the security price after a specified number of ups and down movements is the same, irrespective of the order in which the movements occurred. **[1]**
In a non-recombining binomial tree the values of u and d can change at each stage. As a result, each node in the tree will in general generate two new nodes, making the tree much larger than a combining tree. Consequently, an n period tree will have 2n rather than n+1, possible states at time n.

[1]
Total [2]

b) The Wilkie model attempts to model the processes generating investment returns for several different types of asset.
It can be used to simulate the possible future development of investment returns eg. as part of an ALM exercise. **[1]**
Although it is primarily statistically based-having been estimated using historical data for the relevant time series involved – it does include some constraints upon the parameter values. These are based upon economic theory and consequently has some features of an econometric model. **[1]**
The key variable is the force of inflation
It is modeled as a first-order autoregressive process with normally distributed innovations that is assumed to be the driving force behind the other variables such as :

- Log of equity dividend yield
- Annual change in the long of dividend income.
- Log of the real yield on index-linked bonds. **[1]**

It is therefore a particular case of a vector autoregressive moving average (VARMA) model. **[1]**

It has a Cascade structure so that the process driving each individual variable can be analysed using trasfer functions and without the used to consider all the other variables in the model.

Total [4]

c) Expected payoff

Let S denote the share price at expiry

So S ~ U (90, 110)

i.e. the PDF of S is $\frac{1}{20}$ for $90 \le S \le 110$

Call Option

The pay-off for the call option will be (S – 105)t. The notation (a-b)t used in this solution is an abbreviation for max(a-b, 0).

So the expected value for this payoff is

$$E_{call} = \int_{105}^{110} (S - 105) * \frac{1}{20} dS$$

we can integrate directly using the substitution x = s –105 to get:

$$E_{call} = \int_{0}^{5} x * \frac{1}{20} dx = \frac{1}{20} * \frac{1}{2} * 5^2 = \frac{5}{8}$$

[2]

Put option

The payoff for the put option will be (105-5)t

So the expected value of this payoff is

$$E_{put} = \int_{90}^{105} (105 - s) * \frac{1}{20} ds$$

using $x = 105 - s$ gives

$$E_{put} = \int_{0}^{15} x * \frac{1}{20} dx = \frac{1}{20} * \frac{1}{2} * 15^2 = 5\frac{5}{8}$$

[2]

Total [4]

G Total [10]