Actuarial Society of India

EXAMINATIONS

June 2005

CT4 (104) – Models
Total Marks - 50

Indicative Solution

Q.1 **(i)**

$$m_{x+t} = \lim_{h\to 0}\frac{1}{h}\times P\left[T_x \le t+h | T_x > t\right]$$

[1]

(ii)

$$f_x(t) = \frac{d}{dt}F_x(t)$$

$$= \frac{d}{dt}P\left[T_x \le t\right]$$

$$= \lim_{h\to 0}\frac{1}{h}\times\left\{P\left[T_x \le t+h\right] - P\left[T_x \le t\right]\right\}$$

[1]

Q.2 **a)**

(i)

$$f_x(t) = \frac{d}{dt}F_x(t) = \frac{d}{dt}\,{}_tq_x = \frac{d}{dt}t.q_x = q_x$$

$$= m_{x+t}\ {}_tp_x$$

Thus

$$m_{x+t} = \frac{q_x}{{}_tp_x} = \frac{q_x}{1-t.q_x}$$

by definition of uniform distribution of deaths. **[1]**

(ii) ${}_tp_x = {}_sp_x . {}_{t-s}p_{x+s}$

Thus

$${}_{t-s}p_{x+s} = \frac{{}_tp_x}{{}_sp_x}$$

$${}_{t-s}q_{x+s} = 1 - {}_{t-s}p_{x+s} = 1 - \frac{{}_tp_x}{{}_sp_x} = \frac{{}_sp_x - {}_tp_x}{{}_sp_x}$$

$$= \frac{1 - sq_x - 1 + tq_x}{1 - sq_x} = \frac{(t-s)q_x}{1-sq_x}$$

[2]

b) $p_{56} = 0.990581$ $p_{57} = 0.989503$ and $p_{58} = 0.988314$

(i) udd: ${}_2p_{56.75} = {}_{0.25}p_{56.75}\times p_{57}\times {}_{0.75}p_{58}$

$$= \frac{p_{56}}{1 - {}_{0.75}q_{56}}\times(1-q_{57})\times(1-0.75*q_{58})$$

$$= \frac{0.990581}{1-0.75\times 0.009419}\times(0.989503)\times(1-0.75\times 0.011686)$$

$$= 0.9785044$$

[2]

(ii) Constant force of mortality:

$m = -\log_e p_x = -\log_e(1-q_x)$

Thus ${}_2p_{56.75} = {}_{0.25}p_{56.75}\times p_{57}\times {}_{0.75}p_{58}$

$$= e^{-0.25\times 0.009464}\times 0.989503\times e^{-0.75\times 0.011755}$$

$$= 0.9785$$

[2]

Total [7]

Q.3 Poisson distribution of claims

Mean = nq and std deviation = $\sqrt{nq}$

Thus 5% confidence interval will be equivalent to 5% of the likely death claim or

$1.96 \times \sqrt{nq} = 0.05 \times nq$

or $nq = \left(\frac{1.96}{0.05}\right)^2$

or $n = \left(\frac{1.96}{0.05}\right)^2 * \frac{1}{0.002}$

$= 768320$

[4]

Q.4

The likelihood for each life is

a)

1	2	3	4	5	6	7
p_{40}	${}_{0.3}p_{40.4}$	${}_{0.6}q_{40.3}$	${}_{0.9}q_{40}$	q_{40}	${}_{0.5}q_{40.5}$	${}_{0.4}p_{40}$

[2]

The total likelihood is the product

$(1-q_{40})(1-{}_{0.3}q_{40.4})({}_{0.6}q_{40.3})({}_{0.9}q_{40})(q_{40})({}_{0.5}q_{40.5})(1-{}_{0.4}q_{40})$

$= (1-q_{40}) \times \left[1 - \frac{0.3q_{40}}{1-0.4q_{40}}\right] \times \left[\frac{0.6q_{40}}{1-0.3q_{40}}\right] \times 0.9q_{40} \times q_{40} \times \left[\frac{0.5q_{40}}{1-0.5q_{40}}\right] \times [1-0.4q_{40}]$ **[2]**

$= \frac{(1-q_{40})(1-0.7q_{40})(1-0.4q_{40}) \times 0.27q_{40}^4}{(1-0.4q_{40})(1-0.3q_{40})(1-0.5q_{40})} = \frac{(1-q_{40})(1-0.7q_{40}) \times 0.27q_{40}^4}{(1-0.3q_{40})(1-0.5q_{40})}$

b) The likelihood for each life is proportional to, assuming constant force of mortality m'_{40}

1	2	3	4	5	6	7
$e^{-m'_{40}}$	$e^{-0.3m'_{40}}$	$e^{-0.4m'_{40}}m'_{40}$	$e^{-0.6m'_{40}}m'_{40}$	$e^{-0.8m'_{40}}m'_{40}$	$e^{-0.4m'_{40}}m'_{40}$	$e^{-0.4m'_{40}}$

Thus the total likelihood is the product

$L \propto e^{-3.9m'_{40}}(m'_{40})^4$ **[3]**

Differentiating

$\frac{\partial L}{\partial m'_{40}} = -3.9(m'_{40})^4 e^{-3.9m'_{40}} + e^{-3.9m'_{40}} 4m'^3_{40}$

Equating to zero

$-3.9m'_{40} + 4 = 0$

or

$m'_{40} = 1.0256$

[3]

Total [10]

Q.5

(i) Partial likelihood is

$$L(b)=\prod_{j=1}^{k}\frac{\exp(b\,x_j^T)}{\sum_{i\in R(tj)}\exp(b\,x_i^T)}$$

where k is the number of deaths assumed to occur at distinct times

t_j is the t[th] lifetime

$R(t_j)$ denotes the set of lives at risk at time t_j. **[1]**

(ii) The partial likelihood is

$$\frac{e^b}{7+7e^b}\times\frac{e^b}{6+6e^b}\times\frac{1}{6+3e^b}\times\frac{1}{4+2e^b}\times\frac{e^b}{2+2e^b}\times\frac{1}{1+e^b}$$

$$=\frac{e^{3b}}{504\left(1+e^b\right)^4\left(2+e^b\right)^2}$$

[5]

(iii)

$$L=\frac{e^{3b}}{504\left(1+e^b\right)^4\left(2+e^b\right)^2}$$

$\log L = 3b - 4\log(1+e^b) - 2\log(2+e^b) + \text{const}$

Differetiating with respect to b, we get

$$\frac{d\log L}{d\,b}=3-\frac{4e^b}{1+e^b}-\frac{2e^b}{2+e^b}$$

setting this equal to zero and rearranging ($y = e^b$)

$$\frac{4y}{1+y}+\frac{2y}{2+y}=3$$

or $3y^2+y=6$

Thus $y=1.257$ using only +ve value

$b = \log y = 0.2287$ **[5]**

(iv) We need to use Breslow approximation to get the new partial likelihood which is

$$\frac{e^b}{8+8e^b}\times\frac{e^b}{7+7e^b}\times\frac{1}{7+3e^b}\times\frac{1}{5+2e^b}\times\frac{e^b}{3+2e^b}\times\frac{1}{(2+e^b)^2}$$

[3]

Total [14]

Q.6 a)

(i) Policy year rate interval. **[1]**

(ii) Assume birthdays uniformly distributed over the policy year.

Lives will be thus on the average $x+\frac{1}{2}$ at the start of the rate interval (for q_x estimate)

Thus x + f = x + 1, age at the middle of the rate interval (for m_x estimate)

[2]

(iii) The assumption of uniform birthdays over the policy year now no longer holds.

Lives are now $x+1-\frac{1}{4}= x+\frac{3}{4}$ at the start of the rate interval (q_x type) and $x+1\frac{1}{4}$ at the middle of the rate interval (m_x type). **[2]**

b)

(i) $P_{x,t}$ = number of policyholders in force at 1.1.2000+t aged x nearest birthday on previous policy anniversary.

Company I

Let $P'_{x,t}$ = number of policyholders in force at 1.1.2000+t aged x nearest birthday.

$$E_x^e = \int_0^1 P'_{x,t}\,dt = \frac{1}{2}\left(P'(x,0) + P'(x,1)\right)$$ **[3]**

assuming $P'_{x,t}$ varies linearly over the calendar year

Since deaths are recorded as age nearest birthdays at death.

$$P'(x,t) = \frac{1}{2}\left(P(x-1,t) + P(x,t)\right)$$

assuming policy anniversaries are uniformly distributed over the calendar year and birthdays are uniformly distributed over the policy year.

Thus

$$E_x^e = \frac{1}{4}\left[P(x-1,0) + P(x,0) + P(x-1,1) + P(x,1)\right]$$

Company II

Similarly

$$E_x^e = \int_0^1 P'(x,t)\,dt = \frac{1}{2}\left(P'(x,0) + P'(x+1,1)\right)$$

where $P'(x,t)$ is the number of policyholder in force at 1.1.2000+t aged x nearest birthday on 1.1.2000.

Since deaths are recorded as age nearest birthday on 1.1.2000

$P'(x,t) = \frac{1}{2}\left(P'(x-1,t) + P'(x,t)\right)$ with the same assumptions as the company I.

Thus $E_x^e = \frac{1}{4}\left[P(x-1,0) + P(x,0) + P(x,1) + P(x+1,1)\right]$

[3]

Total [6]

(ii) For company I, the above observed rates apply to age x $-\frac{1}{2}$ for q_x and x for m_x. No assumption is required.

For company II, the observed rates apply to age x for q_x and x+$\frac{1}{2}$ for m_x.

Assumption: Birthdays are uniformly distributed over the calendar year. **[2]**

Total [13]

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