Actuarial Society of India

EXAMINATIONS

June 2005

CT4 (103) – Models
Total Marks - 50

Indicative Solution

Q.1

(i)

a) Let U denote the process described by 2301 and V denote the process described by 2401. Then

$$P[U=6]P[V=0]=\left(\frac{e^{-5}5^6}{6!}\right)\left(e^{-1}\right)=0.0538$$

b) The arrivals of either train will be Poisson with parameter 6 per hour. In half an hour the number of arrivals will have a Poisson (3) distribution. Thus,

$$P[U+V\geq 3]=1-P[U+V=0]-P[U+V=1]-P[U+V=2]$$

$$=1-e^{-3}-3e^{-3}-\frac{e^{-3}3^2}{2!}=0.5768$$

c) The waiting time T till the next is exponential with parameter 6. It is irrelevant whether one has just missed a train or not since the exponential distribution has the memoryless property. Thus, the required probability is:

$$P\left[T>\frac{1}{4}\right]=e^{-\frac{6}{4}}=0.2231$$

d) The distribution of the number of 2301's in time t is Poisson (5t). Thus the probability of exactly one is $e^{-5t}5t.$ However, we don't know the time taken for the 2401 to arrive. The distribution of this is exponential with parameter 1. Thus, we condition on this time. As the time is continuous, we need to use the density function:

$$P\left[\text{exactly one 2301 before a 2401 arrives}\right]=\int_0^\infty P\left[\text{exactly one 2301 in time t}\right]e^{-t}dt$$

$$=\int_0^\infty e^{-5t}5dte^{-t}dt=\frac{5}{36} \qquad \text{(integrated by parts)}$$

[10]

(ii) As 1 in every 6 trains is on average a 2401, the answer is $\frac{1}{6}$. Alternatively, let the waiting time for a 2301 be denoted by T_{2301} and that for a 2401 be denoted by T_{2301}. The probability of a 2401 arriving before a 2301 is then

$P[T_{2401} < T_{2301}]$.

Conditioning on the 2401 waiting time, we thus have

$$P[T_{2401} < T_{2301}] = \int_0^\infty e^{-t}P[\, T_{2301} > t]\,dt$$

$$=\int_0^\infty e^{-t}e^{-5t}dt$$

$$=\frac{1}{6}$$

[2]

(iii) The distribution of the number of Chittaranjan manufactured trains in time it t is Poisson with parameter $\left(\frac{1}{2}+5*\frac{1}{3}\right)t=\frac{13}{6}t$. The waiting time will be thus be exponential with parameter $\frac{13}{6}$. Since the exponential distribution has the lack of memory property, that a Chittaranjan manufactured train hasn't come for over an hour is irrelevant. The expected waiting time is thus $\frac{13}{6}$ hours or about 27.7 minutes.

[3]
Total [15]

Q.2

- A stochastic process X_n is stationary if the joint distributions of the $X_{t_1}, X_{t_2}, \ldots\ldots\ldots, X_{t_m}$ and $X_{t_1+k}, X_{t_2+k}, \ldots\ldots\ldots, X_{t_m+k}$ are identical for all $t_1, t_2, \ldots\ldots\ldots, t_m, k+t_1, k+t_2, \ldots\ldots., k+t_m$ and all integers m.

- The process is weakly stationary of the expectations $E[X_t]$ are constant with respect to t and the covariances $Cov(X_t, X_{t+k})$ depend only on the lag k.

- If t and t+u are in the set of permissible values, then the increment for time u will be $X_{t+u} - X_t$.

- For a discrete process the Markov property requires that:
  $P\left[X_t = x \mid X_{t_1} = x_1, X_{t_2} = x_2, \ldots\ldots\ldots X_{t_m} = x_m\right] = P\left[X_t = x \mid X_{t_m} = x_m\right]$ for

all times $t_1 < t_2 < \ldots.. < t_m < t$ and all states $x_1 < x_2 < \ldots\ldots\ldots < x_m < t$.

- A discrete time martingale X_n satisfies two conditions:
  1. $E[|X_n|] < \infty$ for all n
  2. $E[|X_n| X_0, X_1, \ldots.. X_m] = X_m$ for all m < n.

Total [10]

Q.3 a)

- At each step Aishwarya's fund will change by a random amount Z_n where

  $Z_n = +1$ with probability $\frac{1}{2}$

  $Z_n = -1$ with probability $\frac{1}{2}$

So that $S_n = S_{n-1} + Z_n$ will be a simple symmetric random walk. Initially $S_0 = k$. The boundary conditions are such that

$P\left[S_n = 0 \mid S_{n-1} = 0\right] = 1$ and

$P\left[S_n = K \mid S_{n-1} = K\right] = 1$

So that we can consider it as a walk on (0, 1, 2…..K).

[3]

b)

- Here,

$E[S_n \mid S_0, S_1, \ldots\ldots., S_{n-1}] = E[S_{n-1} + Z_n \mid S_0, S_1, \ldots\ldots., S_{n-1}]$

$= S_{n-1} + E[Z_n] = S_{n-1}$

if $0 < S_{n-1} < K$. In the case where $S_{n-1} = 0$ or $S_{n-1} = K$, the above property holds since then $S_n = S_{n-1}$.

[2]

c)

- A stopping time for the process S_n is a positive valued integer random variable such that for all n, the indicator variable $I_{\{T=n\}} = 1$ if T = n and $I_{\{T=n\}} = 0$ otherwise is a function of the past and present values $S_0, S_1, \ldots\ldots. S_n$ only.

The optimal stopping theorem states that E[S_T] = E[S_0] if either

1. T is bounded, i.e. $T \le N$ for some constant N
2. S is bounded, i.e. $|S_n| \le N$ for some constant N

[4]

d)

- Let T be the stopping time until the process reaches 0 or K. Since S_n is a martingale and bounded, the optimal stopping theorem can be applied. This results in E[S_T] = E[S_0] = k.

  Thus,

  $E[S_T] = 0.P[S_T = 0 \mid S_0 = k] + K.P[S_T = K \mid S_0 = k]$

  $= K(1 - P[S_T = 0 \mid S_0 = k])$

  Since the probability of ruin is just $P[S_T = 0 \mid S_0 = k]$, and we know that $E[S_T] = k$, we can thus obtain $P(S_T = 0 \mid S_0 = k) = \dfrac{K-k}{K}$

[3]

e)

- If Aishwarya is greedy then K is very large. If we get $K \to \infty$ the,

  $P(S_T = 0 \mid S_0 = k) = \dfrac{K-k}{K} \to 1$

In other words, she is certain to be ruined if she does not quit. **[2]**

f)

- Vivek needs to be super rich so that he will not go broke. The stopping time therefore only depends on Aishwarya's position. **[1]**

Total [15]

Q.4

a)

$B_1 \sim N(0,1)$ and $B_2 \sim N(0,2)$. Thus,

$P(B_1 \ge 1) = 1 - \Phi(1) = 0.159$

$P(B_2 \ge 1) = 1 - \Phi\left(\dfrac{1}{\sqrt{2}}\right) = 0.240$

[3]

b)

$B_2 \sim N(0,2)$ Thus

$\mathrm{P}(\mathrm{Z>z}) = P[Z > z, B_2 > z] + P[Z > z, B_2 < z] = 2P[Z > z, B_2 > z]$

$= 2P[B_2 > z]$

Now $\mathrm{P}(\mathrm{Z>z}) = 2\left[1 - \Phi\left(\dfrac{z}{\sqrt{2}}\right)\right]$

$f(z) = 2\dfrac{\partial}{\partial z}\left[1 - \Phi\left(\dfrac{z}{\sqrt{2}}\right)\right] = \dfrac{2}{\sqrt{2}}\Phi\left(\dfrac{z}{\sqrt{2}}\right) = \dfrac{1}{p} e^{-\frac{z^2}{4}}$ z > 0.

[4]

c)

$E(X_T) = E(X_0)$ if T is the stopping time and X_t is a martingale. B_t is a martingale.

Thus, $E(B_{Tx}) = E(B_0) = 0$ for all x.

T_x is the first time the process X_t hits x. Thus $X_{Tx} = x$. Thus

$B_{Tx} + mT_x = x$

$B_{Tx} = x - mT_x$

$E(B_{Tx}) = 0 \Rightarrow E(x - mT_x) = 0$

$\Rightarrow E(T_x) = \dfrac{x}{m}$

[3]

Total[10]

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