Actuarial Society of India

EXAMINATIONS

June 2005

CT1 – Financial Mathematics

Indicative Solution

Question 1

a. Rate of interest over and above the rate of inflation is called "real rate of interest". [1]

b. Real rate of interest will be lower than money rate of interest, when rate of inflation is positive. [1]

c. If "r" is the real rate of return, "e" the inflation and "i" the money rate of return then

$$(1 + i) = (1 + e) (1 + r)$$

[2]

d. Estimate price of the item is $30 * ((1.05)^{0.5})$ = Rs. 30.74 [2]

e. (i) Maturity value as on 01st July 2005 is $1000 * ((1.06)\text{^}0.5)$ = Rs.1029.56

(ii) Let "r" be the real rate of return per annum.

Equation of value: $1000[(1+r)^{0.5}(1.05)^{0.5}] = 1029.563$

Solving for "r", we get r = 0.9524%

[2]

Question 2

Sub section (a)

Accumulation from t = 0 to t = 10 is $150e^{0.04*10}$ = 223.77370

Accumulation from t = 10 to t = 20 is $223.77370 * e^{\int_{10}^{20} 0.001(t-10)^2 + 0.04dt}$

$$= 223.77370 * e^{0.733333}$$

$$= 465.90$$

Thus, the accumulation of Rs 150 at t = 20 is Rs 465.90.

(4 marks: 1 mark for accumulation from 1 to 10;
2 marks for equation for accumulation from 10 to 20
1 mark for correct final answer.)

Sub section (b)

Equation for the present value of a continuous payment stream of Rs 10 between time t = 5 and t = 10 is $\int_5^{10} 10e^{-0.04t}\,dt = 10*\left[\frac{e^{-0.04t}}{0.04}\right]_5^{10}$ = 37.103

(3 marks: 2 marks for the correct equation and 1 mark for correct numerical answer)

Question 3

"i", the effective rate of interest = 0.04

Equation for finding the accumulated value:

$$\frac{100S_{12}}{S_2}(1.04)^{12} + 100S_{12} = 2681.835$$

[Alternate method would be to calculate effective annual rate of return and use it to accumulate the annuity at that rate of return].

[4 marks]

Question 4

Value of Share = $\frac{2.5(1.03)}{\sqrt{(1.08)}} + \frac{2.5(1.03)^2}{(\sqrt{(1.08)})^2} + \frac{2.5(1.03)^3}{(\sqrt{(1.08)})^3} + \ldots$

Assuming $v = \frac{1.03}{\sqrt{(1.08)}}$, corresponding rate of interest, "i" = 0.8962% and the equation simplifies to $2.5 * a_{\infty}$ at "i", and the value of share is then equal to Rs 278.97.

[4 marks]

Question 5

Sub section (a)

Initial amount of the loan = Present value of all loan repayments at appropriate rate of interest.

Thus, initial amount of loan = $1000*[a_{10}^{5\%} + v_{5\%}^{10} * a_{10}^{7\%}]$
= Rs.12033.60

[2 marks]

Sub section (b)

Let the flat rate of interest be “i”.

Equation of value at flat rate of interest “i” is $1000 * a_{20}^{i} = 12033.60$

Solving for “i”, we find that the flat rate of interest per annum to be 5.4165%

[3 marks]

Question 6

Subsection (a) (i)

Monthly repayment under fixed interest basis = $\frac{10000}{a_{24}}$ at monthly equivalent rate of interest for 8% p.a.

Thus monthly repayment = Rs 451/-

(2 marks)

Subsection (a) (ii)

Monthly repayment can be found out using the generic formula :

$\frac{Loan}{Annuityfactor}$, where the numerator would be the loan outstanding on the recalculation date and the annuity factor would be based on the term outstanding and relevant interest applicable then.

Thus monthly repayment for the first six months would be $\frac{10000}{a_{24}}$, where the annuity would be calculated at the monthly equivalent of 7.75% p.a. and is equal to Rs 449.94.

Loan outstanding on the next recalculation date (01st Jan - note that that the monthly repayment calculated on 01st Jan would be payable from 01st Feb) can be calculated using formula $10000(1.0775)^{0.5} - 449.94 S_{6}$ (where the accumulation factor would be calculated at monthly equivalent of 7.75% p.a).

The other approach would be to draw a monthly cash flow table as below and calculate loan outstanding after every 6 monthly payments and use it to calculate revised monthly instalment for the next 6 months.

	Loan O/S (previous month beg)	Interest rate p.a.	Interest rate p.m.	Interest payable on loan	Monthly repayment	Capital repaid	Loan o/s
Aug-05	10000.0000	7.75%	0.00624	62.3968	449.9396	387.5428	9612.4572
Sep-05	9612.4572	7.75%	0.00624	59.9787	449.9396	389.9610	9222.4962
Oct-05	9222.4962	7.75%	0.00624	57.5454	449.9396	392.3942	8830.1020
Nov-05	8830.1020	7.75%	0.00624	55.0970	449.9396	394.8426	8435.2594
Dec-05	8435.2594	7.75%	0.00624	52.6333	449.9396	397.3063	8037.9532
Jan-06	8037.9532	7.75%	0.00624	50.1543	449.9396	399.7854	7638.1678
Feb-06	7638.1678	8.00%	0.00643	49.1442	450.7511	401.6069	7236.5609
Mar-06	7236.5609	8.00%	0.00643	46.5603	450.7511	404.1908	6832.3701
Apr-06	6832.3701	8.00%	0.00643	43.9597	450.7511	406.7914	6425.5787
May-06	6425.5787	8.00%	0.00643	41.3424	450.7511	409.4087	6016.1700
Jun-06	6016.1700	8.00%	0.00643	38.7082	450.7511	412.0429	5604.1271
Jul-06	5604.1271	8.00%	0.00643	36.0571	450.7511	414.6940	5189.4331
Aug-06	5189.4331	8.50%	0.00682	35.3997	451.8666	416.4669	4772.9663
Sep-06	4772.9663	8.50%	0.00682	32.5588	451.8666	419.3078	4353.6585
Oct-06	4353.6585	8.50%	0.00682	29.6985	451.8666	422.1681	3931.4904
Nov-06	3931.4904	8.50%	0.00682	26.8186	451.8666	425.0479	3506.4425
Dec-06	3506.4425	8.50%	0.00682	23.9192	451.8666	427.9474	3078.4951
Jan-07	3078.4951	8.50%	0.00682	20.9999	451.8666	430.8666	2647.6285
Feb-07	2647.6285	8.25%	0.00663	17.5484	451.5643	434.0159	2213.6125
Mar-07	2213.6125	8.25%	0.00663	14.6718	451.5643	436.8926	1776.7200
Apr-07	1776.7200	8.25%	0.00663	11.7760	451.5643	439.7883	1336.9317
May-07	1336.9317	8.25%	0.00663	8.8611	451.5643	442.7032	894.2285
Jun-07	894.2285	8.25%	0.00663	5.9269	451.5643	445.6374	448.5911
Jul-07	448.5911	8.25%	0.00663	2.9732	451.5643	448.5911	0.0000

Thus the monthly instalments and loan outstanding are:

Calculation date	Loan O/S	Monthly instalment	Payable between
Jul-05	10000.00	449.94	from Aug 05 to Jan 05
Jan-06	7638.17	450.75	from Feb 06 to Jul 06
Jul-06	5189.43	451.87	from Aug 06 to Jan 07
Jan-07	2647.63	451.56	from Feb 07 to Jul 07

(Marking schedule: The question requires calculation of 7 values, which are the values given in the table above. Considering the degree of difficulty in calculating these figures the following marking schedule is recommended.

2 marks for correctly calculating monthly instalments at calculation date Jul-06.
3 marks for correctly calculating loan o/s as at calculation date Jan-06.
3 marks for correctly calculating monthly instalments at calculation date Jan-06.
1.5 marks for correctly calculating the remaining 4 values.)

Sub section (a) (iii)

Equation of value:

$$10000 = 449.94a_{6} + 450.75v^{6}a_{6} + 451.87v^{12}a_{6} + 451.56v^{18}a_{6}$$

By trial and error, the annual flat rate of interest is very close to 8.00015%.

(3 marks: 2 for the equation of value and 1 mark for correctly calculation)

Subsection (b)

Since the flat rate of interest is very close to the fixed rate of interest both the options are equivalent. Since the flat rate of interest is very slightly higher than the fixed rate of interest, there are reasons to believe that fixed interest might be a better option (but this could be due to rounding off errors too and hence it is ideal to believe that both the options are equivalent).

To calculate profit from opting fixed interest rate, we need to find the present value of payments that would be expected to be paid under floating rate option at 8% (the fixed rate of interest).

Using the LHS of the equation of value given in subsection (a) (iii), the present value of payments that is expected to payable under floating rate option at 8% p.a is Rs.10000.01. Thus the expected profit from opting for fixed rate of interest is Rs.0.01.

(Students are expected to clearly provide the method of calculation of profit from the chosen method to get full marks – mere indication of equality of flat rate and fixed rate of interest will not fetch full marks).

(4 marks: 2 marks for the approach, 1 mark for calculating the profits and 1 for stating with appropriate reasons, which basis is better.)

Question 7

Sub section (a)

Let the price of the fixed interest security be A.

Thus A $= 10.0 * 0.75 * a_{20}^{(2)} + 110v^{20}$ at 10%

= (7.5 * 8.7214) + (110 * 0.148644)
= 81.761 or Rs 81.76%

(6 marks – 4 for the equation and 2 for correct numerical value)

Sub section (b)

Let volatility of the fixed interest security be D.

Thus D = $\dfrac{(7.5/2)*(0.5v^{1.5}+v^{2}+1.5v^{2.5}+...+20v^{21})+20*110*v^{21}}{81.761}$

Using fundamental equation solving techniques, the following can be deduced.

$$0.5v^{1.5}+v^{2}+1.5v^{2.5}+...+20v^{21} = 2v*\frac{\ddot{a}_{20}^{(2)}-20v^{20}}{i^{(2)}} = 114.998$$

Thus D = 8.910

(5 marks – 3 for the equation, 2 for deriving correct numerical value)

Question 8

$$\text{DMT} = \frac{v+2v^{2}+3v^{3}+...+10v^{10}}{v+v^{2}+v^{3}+...+v^{10}}$$

$$= \frac{(Ia)_{10}}{a_{10}} \text{ at } 7\%$$

$$= \frac{34.7393}{7.0236}$$

= 4.946 years

(5 marks – 2 for the equation, 3 for deriving correct numerical value)

Question 9

Present value of dividends = $0.3v_{4\%}^{1/2}+0.3v_{4.5\%}$
= 0.3 × (0.980581 + 0.956938)
= 0.581256

Forward price of the share = (6 – 0.581256) * (1.045) = Rs. 5.66259

(5 marks: 1 mark for the approach, 3 for the equation, 1 for correct calculation)

Question 10

Sub section (a)

An agreement where two parties exchange fixed and floating rate of interest. One party agrees to pay a floating rate and receive a fixed interest rate and the other party agrees to pay a fixed interest rate and receive a floating interest rate. Both sets of payments are in the same currency.

(2 marks)

Sub section (b)

The fixed payments are at <u>a constant rate</u> for <u>an agreed term</u> and the floating payments will be <u>linked</u> to the level of a <u>short-term interest rate</u>.

(2 marks - 1/2 mark for each of the terms underlined)

Sub section (c)

Each counterparty faces market and credit risk.

Market risk: The risk that market conditions will change so that the present value of the net outgo under the agreement increases.

Credit risk: The risk that the counterparty will default on its payments. This will occur only if the swap has a negative value to the defaulting party.

(3 marks – 1 mark for market risk and 1 each for the two points in credit risk)

Question 11

Term structure of interest rates:

Definition: The variation by term of interest rates is referred to as the term structure of interest rates.

Three popular theories that explain the term structure of interest rates:

- Expectations Theory
- Liquidity preference theory
- Market segmentation theory

(2 marks: ½ mark each for the definition and naming the three theories)

Question 12

Let m be the one-year spot rate and n be the two-year spot rate.

Let A = $\frac{1}{(1+m)}$ and B = $\frac{1}{(1+n)^2}$

For the two-year fixed interest stock we have

105.40 = 8A + (8+98)B --------------------------------- (1)

From the information on two-year par yield we have

100 = 4.15A + (100+4.15)B ---------------------------- (2)
(Please refer to the note at the end of this solution.)

Solving (1) and (2),

A = 0.959601 and B = 0.921917

Thus m = 4.21% and n = 4.15%

(Note that there had been a typo in the question paper in which the 2 year par yield is given as 5.25% instead of 4.15%, which leads to negative yields. In such a case, m = -42.13% and n = 7.59%. Thus, full mark to be awarded for students who have arrived at this solution too – provided the approach adopted by them is reasonable.)

(6 marks: 2 marks each for the two equations and 2 marks for solving the equations)

Question 13

Students are expected to state the following three conditions which are to be satisfied for immunization.

Present value of Assets and Liabilities should be equal
Discounted mean term of Assets and Liabilities should be equal
And convexity of assets > convexity of liabilities.

(2 marks: ½ mark for each point and ½ mark for clarity)

Question 14

$E[i_t] = 0.08 * 0.625 + 0.04 * 0.25 + 0.02 * 0.125$
= 0.0625 = j (say)

$\therefore E[S_3] = (1+j)^3 = 1.0625^3 = 1.1995$

$V[i_t] = 0.08^2 * 0.625 + 0.04^2 * 0.25 + 0.02^2 * 0.125 - 0.0625^2$
$\quad = 0.000544 = s^2$ (say)

Using the notations above,

$V[S_n] = ((1+j)^2 + s^2)^n - (1+j)^{2n}$

Thus $V[S_3] = ((1.0625)^2 + 0.000544)^3 - (1.0625)^6 = 0.002081$

Standard deviation of S_3 = 0.04562

(5 marks: 2 marks for expected value and 3marks for standard deviation)

Question 15

Given that (1+i) is log normally distributed with mean 1.0015 and variance $9*10^{-6}$.

To derive the parameters of the corresponding normal distribution, we have

$$\exp\left(m + \frac{s^2}{2}\right) = 1.0015 \text{ and}$$
$$\exp(2m + s^2)(\exp(s^2) - 1) = 9*10^{-6}$$

Solving these two equations, we have $m = 0.00014944$ and $s^2 = 8.9730*10^{-6}$.

Thus we have, $\ln(1+i)$ follows normal distribution with the mean 0.00014944 and variance $8.9730*10^{-6}$.

We need to find j such that P[i ≤ j] = 0.1

Using the distribution of ln(1+i) we have,

$$P\left(\frac{\ln(1+i) - 0.0014944}{\sqrt{8.9730*10^{-6}}} \le -1.28155\right) = 0.1$$

Re-arranging the terms, we have

P(i ≤ -0.0023417) = 0.1

Thus j = -0.23417%

(6 marks: 4 marks for m and s, and 2 marks for calculating j)