Actuarial Society of India

EXAMINATIONS

14th June 2005 (am)

Subject CT3 – Probability and Mathematical Statistics

Time allowed: Three Hours (10.30 am - 13.30 pm)

INSTRUCTIONS TO THE CANDIDATES

1. *Do not write your name anywhere on the answer scripts. You have only to write your Candidate's Number on each answer script.*

2. *Mark allocations are shown in brackets.*

3. *Attempt all questions, beginning your answer to each question on a separate sheet.*

4. *Fasten your answer sheets together in numerical order of questions. This, you may complete immediately after expiry of the examination time.*

5. *In addition to this paper you should have available graph paper, Actuarial Tables and an electronic calculator.*

Professional Conduct:

"It is brought to your notice that in accordance with provisions contained in the Professional Conduct Standards, If any candidate is found copying or involved in any other form of malpractice, during or in connection with the examination, Disciplinary action will be taken against the candidate which may include expulsion or suspension from the membership of ASI."

AT THE END OF THE EXAMINATION

Hand in both your answer scripts and this question paperto the supervisor.

Q.1 **a)** The following data represent the scores obtained by 27 students in a midterm test

79	78	78	67	76	87	85	73	66
99	84	72	66	57	94	84	72	63
51	48	50	61	71	82	93	100	89

    i) Prepare ste m and leaf display of the data **[2]**

    ii) Calculate the range and the interquartile range. **[2]**

b) The ages in years of a group of 20 individuals are as follows:

50	56	55	49	52	57	56	57	56	59
54	55	61	60	51	59	62	52	54	49

If $\bar{x}$ denotes the mean and s the standard deviation of the sample, find the percentage of items falling within the interval $\bar{x} \pm s$.

(Given $\sum x = 1104$ $\sum x^2 = 61226$) **[3]**

Total [7]

Q.2 **a)** State and prove Bayes' Theorem **[3]**

b) The random variables (X,Y) have the following joint probability mass function

Y \ X	1	2	3	
2	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{12}$	$\frac{1}{3}$
3	$\frac{1}{6}$	0	$\frac{1}{6}$	$\frac{1}{3}$
4	0	$\frac{1}{3}$	0	$\frac{1}{3}$
	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	

    i) Show that X and Y are dependent. **[2]**

    ii) Give a probability table of random variables U and V that have the same marginals as X and Y but are independent. **[3]**

Total [8]

Q.3 **a)** The probability that there is at least one error in an account statement prepared by A is 0.2 and for B and C, the probability is 0.25 and 0.4 respectively. A,B and C prepared 10,16 and 20 statement respectively. Find the expected number of correct statements in all. **[3]**

b) Let Z have a triangular density function

$$g(z) = \begin{cases} z & 0 \le z \le 1 \\ 2 - z & 1 \le z \le 2 \\ 0 & \text{elsewhere} \end{cases}$$

    i) Derive the mgf of Z **[3]**

    ii) Let X_1 and X_2 be independent uniform random variables on the interval (0,1), that is each with density $f(x) = 1$ $0 < x < 1$

                                                      $= 0$ elsewhere

    Using the definition and properties of mgf, derive the mgf of $Y = X_1 + X_2$ and hence comment on g(z). **[3]**

Total [9]

Q.4 **a)** An Insurance company found that only 0.01% of the population is involved in a certain type of accident each year. If its 1000 policyholders can be regarded as randomly selected from the population, what is the probability that not more than two of its clients are involved in such accidents? **[3]**

b) Define Poisson Process and show that the interarrival time follows an exponential distribution. **[4]**

c) If two independent random variables X and Y have the same geometric distribution, prove that the conditional distribution of X given that $X + Y = k$ (constant), is a discrete uniform distribution. **[4]**

Total [11]

Q.5 **a)** State the central limit theorem for independent identically distributed random variables.

In a large population the distribution of a variable has mean 167 and standard deviation 27 units.

If a random sample of size 36 is chosen, find the approximate probability that the sample mean lies between 163 and 171 units. **[3]**

b) Consider a sphere of radius X, where X has the distribution function

$$F(x) = \begin{cases} 0 & x < 0 \\ 3x^2 - 2x^3 & 0 \le x \le 1 \\ 1 & x \ge 1 \end{cases}$$

The volume V of the sphere of radius X is $V = \frac{4}{3}p X^3$ cubic units.

Find the probability that the volume of the sphere is greater than V_0.
For what values of V_0 is the above probability zero? **[4]**

Total [7]

Q.6 Let $x_1, x_2, \ldots x_n$ be a random sample of size n from the exponential distribution with probability density function

$$f(x) = \frac{1}{\ddot{e}} e^{-\frac{x}{I}} \quad x > 0, I > 0$$

i) Show that the MLE of I is given by

$$\hat{I} = \frac{\sum x_i}{n}$$

[3]

ii) Show that $\hat{I}$ is an unbiased estimator of I. **[1]**

iii) Obtain the Cramer – Rao lower bound for the variance of the unbiased estimators of I. **[3]**

iv) Show that the variance of $\hat{I}$ attains the Cramer – Rao lower bound in (iii). **[1]**

v) Obtain the 95% approximate confidence interval for I. **[2]**

Total [10]

Q.7 A social worker interested in reducing obesity among children visited a village and recorded the weight (in kg) of 20 children.
The following are the weights of these 20 children

65	62	70	62	64	72	55	50	60	60
70	64	55	63	64	70	56	54	50	69

$$\sum x = 1235, \qquad \sum x^2 = 77117$$

a) Plot the data using a dot plot and comment on whether measurements follow normal distribution. **[1]**

b) Calculate 95% confidence interval for the mean weight of obese children in the village. **[5]**

During the visit, the social worker advised the children to have routine physical exercises and prescribed diets to reduce the weights. After two months the social worker revisited the village and recorded the weights of the same children. The data of recorded weights are given below in the same order of the children.

| 62 | 60 | 68 | 62 | 60 | 68 | 60 | 50 | 58 | 61 |
| 68 | 64 | 56 | 62 | 60 | 68 | 55 | 51 | 55 | 70 |

c) Carry out an appropriate t – test to investigate whether the advice given to the children is effective in reducing the weights. **[5]**

Total [11]

Q.8 A bird watcher sitting in a park has spotted a number of birds belonging to six categories. The exact classification is given below

Category	1	2	3	4	5	6
Frequency	6	7	13	14	9	5

Test at 5% level of significance whether or not the data are compatible with the assumption that the park is visited by the same proportion of birds belonging to these six categories in the population. **[4]**

Q.9 The following data relate x, the moisture of wet mix of a certain product and y, the density of the finished product.

x	12	11	10	9	8
y	4	3	2	0	1

a) Draw a scatter plot and comment. **[1]**

b) Calculate the coefficient of correlation **[3]**

c) Assuming a linear relationship of the form $Y = b_0 + b_1 X + \text{error term}$,

Obtain the least squares estimate of b_0 and b_1. **[4]**

d) Find the standard error of estimate. **[2]**

e) Obtain the 95% confidence interval for b_1, stating the assumptions on the error term. **[3]**

f) In testing $H_0 : b_1 = 0$ against $H_1 : b_1 \neq 0$, state with reason whether H_0 can be accepted at the 5% level of significance. **[2]**

Total [15]

Q.10 Three training methods were compared to see of they led to greater productivity after training. Below are productive measures for individuals trained by each method.

Method 1	36	26	31	20	34
Method 2	40	29	38	32	39
Method 3	32	18	23	21	33

Test whether the three methods lead to different levels of productivity. Use the 5% level of significance. (Given $\sum \sum y_{ij}^2 = 14326$) **[6]**

Q.11 a) Let(X,Y)have the joint density

$$f(x,y) = \begin{cases} xe^{-x(1+y)} & x \geq 0 \quad y \geq 0 \\ 0 & \text{elsewhere} \end{cases}$$

Show that

i) E(Y) does not exist **[2]**

ii) $E(Y/X) = \dfrac{1}{x}$ x>0 **[2]**

b) i) Define a compound distribution **[2]**

ii) Let $S = \sum_{i=1}^{N} X_i$, where N is a Poisson variable with mean 3 and X_i's are independent identically distributed binomial variables with parameters n = 15 and p = $\dfrac{1}{3}$.

Find the expected value and variance of S. **[6]**

Total [12]

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