Institute of Actuaries of India

Subject ST6 – Finance and Investment B

May 2013 Examinations

INDICATIVE SOLUTIONS

1. i) To prove they are equal we need to create two portfolios where pay-offs are identical. Let's assume that a future contracts last for n days, F_i is the futures price at the end of day i, ($0 < i <= n$), δ is the risk-free rate per day, and on maturity (day n) the stock price is ST.

To show equivalence we need to create two portfolios with identical pay-off and show that there price at time 0 should be equal. We start at the end of day 0 and go long for e^{δ}future contracts and increase the same for e^{δ}number of contracts every day till n. So, by beginning of day i we will have $e^{i\delta}$contracts. Therefore the change in the margin account on day i is the change in futures price times the size of our position at the start of the day, which equals:
$(F_i - F_{i-1})e^{i\delta}$. This change rolled up with interest till day n at the same risk free rate would be
$(F_i - F_{i-1})e^{i\delta}e^{(n-i)\delta} = (F_i - F_{i-1})e^{n\delta}$
Summing over all n days would give $\sum_{i=1}^{i=n}(F_i - F_{i-1})e^{n\delta} = (F_n - F_0)e^{n\delta} = (S_T - F_0)e^{n\delta}$
Additionally, we also invest F_0in a risk free bond till n days and hence the total on maturity of the above strategy would be $(S_T - F_0)e^{n\delta} + F_0e^{n\delta} = S_Te^{n\delta}$. Therefore, this strategy would require an initial outlay of F_0 and would payoff $S_Te^{n\delta}$ on maturity.

Now create an alternative portfolio using forward. Suppose the forward price at the end of day 0 is G_0 . Strategy 2 is to take a long position of $e^{n\delta}$ contracts in end forwards at the start of the contract (the end of day 0) and to invest G_0 in a risk-free zero-coupon bond. By the end of day n (ie time T) the value of the bond will just be sufficient to pay the delivery price for the forwards. So, overall, this strategy requires an initial investment of G_0 and gives $S_Te^{n\delta}$ on maturity.

Since both the strategies/portfolios give identical pay-offs they should have identical starting values i.e. $F_0=G_0$ or future price = forward price

(5)

ii) If risk free rate is not deterministic then there is a possibility that interest rate may be correlated with the underlying (say equity) which would either increase the value of future or decrease. For example, say they are negatively correlated, so when a long future contract makes a gain, the interest would be lower and investment gain would be lower where as if the long future makes a loss the interest rate would go up and the funding would be at a higher rate. Therefore in either situation, the investment gains are lower for the futures (for negative correlation) in comparison to forward reducing attractiveness of futures contract. Hence, they are not likely to be equal.

(3)
[Total 8 Marks]

2. **i)** We use the identity $2ab = (a+b)^2-(a^2+b^2)$. Now we may use Ito lemma to right hand side components of this equation and recognizing that the sum also follows the stochastic process given below:

$$d(X_t+Y_t) = (\mu_t+\nu_t)dt + (\sigma_t+\rho_t)dW_t$$

$$\underline{d(X_t^2) = 2X_t dX_t + \sigma_t^2 dt}$$

$$d(Y_t^2) = 2Y_t dY_t + \rho_t^2 dt$$

$$d[(X_t+Y_t)^2] = 2(X_t+Y_t)d(X_t+Y_t) + (\sigma_t+\rho_t)^2 dt$$

Putting the above together gives us:

$$d(X_tY_t) = \frac{1}{2}d\{(X_t+Y_t)^2 - X_t^2 - Y_t^2\} = X_t dY_t + Y_t dX_t + \sigma_t\rho_t dt$$

(3)

ii) $$X_t - X_s = \frac{1}{\sqrt{\alpha}}(W_{\alpha t} - W_{\alpha s})$$

It can be seen easily that it is continuous and has value zero at time zero since W is a Brownian motion. From the above it can be seen that the increment follows a normal distribution since difference of two normals with mean zero and variance $\frac{1}{\alpha}(\alpha t - \alpha s)$=t-s ie. Same a Brownian motion. Since it has all the properties of a Brownian motion it is a Brownian motion.

$$X_t - X_s = \rho(W_t - W_s) + \sqrt{1-\rho^2}(\widetilde{W}_t - \widetilde{W}_s)$$

It can be easily seen that X is continuous being a sum of two continuous functions and also has a value of zero at time zero as both components would be zero. Also being a linear sum of two normal distributions the distribution of X is also normal. The increment above is independent of any value prior to time s because each of the separate components are independent of any value prior to time s. The mean of the increment is zero as both the components above would be zero. The variance is $\rho^2(t-s) + (1-\rho^2)(t-s) = t-s$. Since all the properties of Brownian motion is satisfied this is a Brownian motion. **(3)**

iii) $$E(X_t) = E\left(\int_0^t W_s^2 dW_s\right) = 0$$

$$V(X_t) = E\left(\int_0^t W_s^2 dW_s\right)^2 = \int_0^t E(W_s^4)ds = \int_0^t 3s^2 ds = t^3$$

(2)

[Total 8 Marks]

3. **i)** If $\widetilde{W}_t = W_t + \int_0^t \gamma_s ds$ where W_t is a standard Brownian motion under the probability measure P, and γ_t satisfies the technical condition that $E_P\left[exp\left(\frac{1}{2}\int_0^T \gamma_s^2 ds\right)\right] < \infty$, then $\widetilde{W}_t$ is a standard Brownian motion under measure Q defined by $\frac{dQ}{dP} = exp\left(-\int_0^T \gamma_s dW_s - \frac{1}{2}\int_0^T \gamma_s^2 ds\right)$ where T is any fixed time horizon.

(1)

ii)

Let the price of f and the numeraire g have the following SDE

$$\frac{df}{f} = \mu_f dt + \sigma_f dz$$

$$\frac{dg}{g} = \mu_g dt + \sigma_g dz$$

Where dz is the increment of the standard Brownian motion under the real world probability measure.

The market price of risk is $\frac{\mu_f - r}{\sigma_f} = \frac{\mu_g - r}{\sigma_g} = \sigma_g$ *which implies*

$$\frac{df}{f} = (r + \sigma_f \sigma_g)dt + \sigma_f dz$$

$$\frac{dg}{g} = (r + \sigma_g^2)dt + \sigma_g dz$$

Using the relationship between SDE of log(f) and f we can obtain the SDE of log(f) and log(g) as follows:

$$d(logf) = \left(r + \sigma_f \sigma_g - \frac{1}{2}\sigma_f^2\right)dt + \sigma_f dz$$

$$d(logg) = \left(r + \sigma_g^2 - \frac{1}{2}\sigma_g^2\right)dt + \sigma_g dz$$

Therefore the SDE of log(f/g) can now be obtained by subtracting one from the other

$$d\left(log\frac{f}{g}\right) = -\frac{1}{2}(\sigma_f - \sigma_g)^2 dt + (\sigma_f - \sigma_g)dz$$

Again using the relationship between a log() and unlogged() SDE we can obtain the differential equation for f/g by adding back the drift as follows:

$$\frac{d\left(f/g\right)}{f/g} = \left[-\frac{1}{2}(\sigma_f - \sigma_g)^2 + \frac{1}{2}(\sigma_f - \sigma_g)^2\right]dt + (\sigma_f - \sigma_g)dz = (\sigma_f - \sigma_g)dz$$

Since this SDE has no drift the discounted asset prices are martingale.

(5)

iii) Let $f = V_t$ *and* $g = N_t$ then applying the result that f/g is martingale from part b to these we get $V_t = N_t E_Q(N_T^{-1} V_T | F_t) = N_t E_Q(N_T^{-1} X | F_t)$ since the value of the derivative at time T is the payoff X.

(1)

iv) A quanto derivative has a payoff that is defined by variables associated with one currency but is actually paid off in another currency.

(1)

v) Let's denote $Y_t = B_t^{-1} C_t D_t$ *and* Z_t *the other one.* Then we have the following SDE for both of them,

$$\frac{dY_t}{Y_t} = 0.06dW_1 + 0.08dW_2 + 1.105dt$$

$$\frac{dZ_t}{Z_t} = 0.36dW_1 + 0.08dW_2 + 2.068dt$$

Therefore, the market price of risk would be such that the drift would go to zero post transformation which can be obtained by solving the following equation

$$\begin{bmatrix} 0.06 & 0.08 \\ 0.36 & 0.08 \end{bmatrix} \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} = \begin{pmatrix} 0.205 \\ 0.268 \end{pmatrix}$$

γ_1 = (0.268-0.205)/0.30 = 0.21

γ_2= (0.205-0.06*0.21)/0.08 = 2.405

(5)

vi) In a CAPM world, the instantaneous returns on the call and the stock may be expressed in terms of their betas as E [dS_t/S_t]= $(r + \beta_S r^m)dt$

$$E\left[{}^{dC}/_{C}\right] = (r + \beta_C r^m)dt$$

where β_S, β_C represent the betas of the stock and the call, r is the risk-free rate, and r^m is the excess expected return on the market portfolio (i.e., the expected return on the market portfolio less the risk-free rate). So clearly, the expected return on the call is greater as long as $\beta_C > \beta_S$. This is an inequality that always holds since the call is akin to a levered position in the stock. Formally, we have

$$\beta_C = \frac{S}{C}\frac{\partial C}{\partial S}\beta_S$$

So, $\beta_C > \beta_S$ as long as $\frac{\partial C}{\partial S} > \frac{C}{S}$. It can be shown that this inequality must always hold as a consequence of the convexity of C in the stock price S. In the specific context of Black Scholes model it is easy to see: the term $\frac{\partial C}{\partial S} = N(d_1)$, so writing the full expression for the call we get

$$\frac{\beta_C}{\beta_S} = \frac{SN(d_1)}{SN(d_1) - PV(K)N(d_2)}$$

(4)
[Total 17 Marks]

4. **i)** The security is more likely to be a put and not a call, because the value of the security increases as j declines, i.e., as the stock price falls.

(1)

ii) In the explicit model, we can solve for the values of the trinomial probabilities, as follows: (v = 0:3, r = 0:1)
$p1 = v^2h/(2k^2) + h(r- 0.5v^2)/(2k) = 0.398$
$p3 = v^2h/(2k^2) - h(r-0.5v^2)/(2k) = 0.352$
p2 = 1 - p1 - p3 = 0:25
V (i; j) = (p1 V (i + 1; j + 1) + p2 V (i + 1; j) + p3 V (i + 1; j - 1) / (1 + rh)
= (0.398(4.5) + 0.25(5.2) + 0.352(6.3))/(1 + 0.1/12) = 5:2647

(4)
[Total 5 Marks]

5. The net return to the short is

Bond	Return
1	114.81 x 1.3987 – 162.63 = -2.04
2	114.81 x 1.2820 – 138.97 =8.22
3	114.81 x 1.1273 – 131.06 =-1.63

Therefore Bond 2 is the cheapest to deliver.

[Total 3 Marks]

6. **i)** V and HW allow the short rate to take negative values, whereas CIR does not.
However, CIR has the problem that it imposes a lower limit on the short rate.
V is the simplest to deal with algebraically. CIR is more complicated.
V is the simplest to deal with statistically because the pricing formulae involve normal distributions. For CIR they involve the non-central chi-square distribution.
V and CIR are both time-homogeneous and therefore lack flexibility but HW does not have this constraint, which makes it more complicated.

(6)

ii) Theoretically, it is possible to reproduce the cash flows associated with the equity using the derivatives. However, there are complications in real world which increases the risk to exact matching. Matching the dividend stream is usually quite difficult. There are certain management actions for which option may not provide any protection like merger/acquisition due to lack of voting rights. Credit risk increases and risk of rebalancing exactly would increase due to indivisibility in real life. For some of the equities suitable options may not be available or may not be liquid. The tax implications may be different giving rise to additional costs. The overall dealing cost may increase due to regular roll-over. The regulations/rules may not permit the same. Underlying interest rates within the options may not be same as that can be earned by the firm.

(6)
[Total 12 Marks]

7. i) The n-th to default contract is one which pays a contractual amount only if n or more bonds in a chosen basket of m, (m > n) bonds defaults. If n = 1, then the value of the contract increases if default correlation across names in the basket decreases. This is because the probability of a single default is higher when all the names are uncorrelated or negatively correlated. In the extreme case when correlation is -1, the fact that one issuer does not default makes it very likely that one of the others will.

If n > 1, then reasonable levels of positive correlation are required to make the required number of defaults occur – so the value of a second to default contract increases as correlation increases. As n increases, greater correlation is better for the buyer of the contract.

(3)

ii) If default risk declines, then all tranches will appreciate in value, though tranche E, being the one that bears first loss, will appreciate more, followed by tranche B, and then tranche A.

When credit correlations increase, tranche E will increase in value. This tranche bears first loss, and is akin to a first-to-default (FTD) basket. As we have seen in part a, FTD risk declines when credit correlations increase. Hence, tranche E will be worth more. Tranches B and A, ceteris paribus, experience greater risk when correlations rise. Overall, declining default risk and increasing credit correlations increase equity (E) tranche values, and have mixed effects for senior (A) and mezzanine (B) tranches.

(3)
[Total 6 Marks]

8. Since the model indicates that convertible bond is under-priced in the market, take a long position in the convertible bond. You are now long equity risk, default risk, and interest rate risk. If you dynamically hedge these away, then the position is risk-less, meaning that it does not change in value with the underlying risks, so that when the price does move to the model value, you can sell of the positions in the convertible bond and the hedges to capture the current price difference between market and model.

What hedges should we add to the position in the convertible bond? First, hedge out the equity risk by shorting stock to reduce the equity position. Then, hedge out the interest rate risk, e.g., by using interest rate futures. Lastly, eliminate default risk using CDS contracts. Since a default would also affect the value of the equity in the hedging portfolio, the amount of CDS purchased would have to be calibrated carefully to reflect this.

[Total 4 Marks]

9. i) We know that the absence of arbitrage implies that the expected normalized prices of assets are martingales if the expectation is taken under the risk-neutral (or martingale) measure. Discounting (normalization) of assets in these cases is undertaken using the money market account, i.e., the spot risk-free asset. This asset is also known as the “numeraire". In contrast, were the numeraire taken to be the forward price of the asset, such as a forward bond, then the probability measure under which normalized asset prices are martingales would be different than when the numeraire is the spot asset. Such a probability measure is known as the “forward measure”.

When pricing interest rate derivatives, it is often more convenient and tractable to work under the forward measure. This is because it results in a separation of the payoff on the derivative from the discount function. This accommodates more complex interest rate models. The idea of the forward measure and its application was first developed by El Karoui, Geman and Rochet in 1985.

(3)

ii) The MTM value on a $1 notional of the FRA at 5.20% is
MTM(5:20) = (0.052 – 0.050) x (183/360) / (1 + 0.052 x 183/360) = 0.00099

The MTM value on a $1 notional of the FRA at 5.21% is
MTM(5:21) = (0.0521 – 0.050) x (183/360) / (1 + 0.0521 x 183/360) = 0.00104

Hence, the PVBP of the FRA is the difference of these two MTM values (which differ from each other by a basis point). The sign of the PVBP is positive, because an increase in rates by 1 bp results in an increases in the MTM value of the FRA. The PVBP is equal to 0.00104 – 0.00099 = 0.00005.

(5)
[Total 8 Marks]

10. i) Since equity is a call option on the firm, we solve the following equation for sigma :
Call option value[V = 100;D = 70; T = 3; sigma; r = 0:04] = 40
which returns sigma = 0:24237 or 24.237%.
The risk-neutral probability of default in the Merton model is given by the formula N(-d_2). First we calculate
d_2 = (ln(V/D) + (r-sigma^2/2)T) / (sigma $T^{0.5}$) = 0:92559
Hence, the probability of default is N(-d_2) = N(-0:92559) = 0:17733 or 17.733%.

(4)

ii) The DTD is the number of standard deviations of current firm value that the firm is away from default. Applying the formula used by KMV is:
DTD = (V – D)/(sigma x V) = (100 – 70) / (0.24237 x 100) = 1:23778 or roughly 1.24 standard deviations from the default point. This is a fairly risky firm from the point of view of default risk.

(2)

iii) At very short debt maturities in the Merton model, there is very low probability of the firm defaulting (there is insucient time to reach default given the diffusion-driven firm value process), so spreads are small. At very long maturities, conditional on the firm surviving the short and medium term, it is likely that the value of the rm will be high|so long term spreads will also be low. At intermediate maturities, firms that are solvent today have sufficient time to run into trouble. Therefore, medium term spreads, not benetting from the effects described in (a) and (b) tend to be higher than short-term or long-term spreads, resulting in a spread curve that is hump-shaped.

(3)
[Total 9 Marks]

11. i) Using Ito's lemma we have,

$$dP = [k(\theta - r)dt + \sigma dB]P_r + \frac{1}{2}\sigma^2 P_{rr}dt + \eta dB\, P_\theta + \frac{1}{2}\eta^2 P_{\theta\theta}dt + \sigma\eta P_{r\theta}dt + P_t dt$$

This may be written with time t replaced by T (time to maturity) as follows:

$$dP = [k(\theta - r)dt + \sigma dB]P_r + \frac{1}{2}\sigma^2 P_{rr}dt + \eta dB\, P_\theta + \frac{1}{2}\eta^2 P_{\theta\theta}dt + \sigma\eta P_{r\theta}dt - P_T dt$$

(3)

ii) Using the pricing relation given in the question E(dP) = rPdt we get,

$$rP = k(\theta - r)P_r + + \frac{1}{2}\sigma^2 P_{rr} + \frac{1}{2}\eta^2 P_{\theta\theta} + \sigma\eta P_{r\theta} - P_T$$

The boundary condition is P(r, θ,T=0) = 1 i.e. at maturity the value of the bond would be 1.

(2)

iii) Using the assumed solution we get

$$P_r = -BP\,, P_{rr} = B^2P\,, P_\theta = -CP\,, P_{\theta\theta} = C^2P, \qquad P_{r\theta} = BCP\,, P_T = \frac{A_T}{A}P - P(rB_T + \theta C_T)$$

Substituting the above in the pde and rearranging so as to separate variables we have:

$$0 = r[kB + B_T - 1] + \theta[-kB + C_T] + \left[\frac{1}{2}\sigma^2B^2 - \frac{A_T}{A} + \frac{1}{2}\eta^2C^2 + BC\sigma\eta\right]$$

Each of the square bracket should be zero which gives three ODE and solution of the same gives us

$$B(T) = \frac{1 - e^{-kT}}{k}, \quad C(T) = T - B(T), \qquad A(T) = exp\left[\int\left(\frac{1}{2}\sigma^2B^2 + \frac{1}{2}\eta^2C^2 + BC\sigma\eta\right)dT\right]$$

(7)

iv) The extra volatility in will make the bond prices higher than in the model with constant mean rate. **(2)**

[Total 14 Marks]

12. Real world default probabilities are the true probabilities of default. They can be estimated from historical data. Risk neutral default probabilities are the probabilities of defaults in a world where all market participants are risk neutral. They can be estimated from bond's prices. Risk neutral default probabilities are higher and returns in the risk neutral world are lower.

The probability of a company moving from AA to A or lower is 6.08%. An estimate of the value of the derivative is therefore 6.08% x 100 x exp(-8%) = 5.61. The approximation in this is that we are using the real world probability of a downgrade. To value the derivative correctly we should use the risk neutral probability of a downgrade. Since the risk neutral probability of a default is higher than the real world probability, it seems likely that the same is true of a downgrade too. This may imply that 5.61 is likely to be too low as an estimate of the value of the derivative.

[Total 6 Marks]

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