Institute of Actuaries of India

Subject CT8 – Financial Economics

May 2013 Examinations

INDICATIVE SOLUTIONS

Introduction

The indicative solution has been written by the Examiners with the aim of helping candidates. The solutions given are only indicative. It is realized that there could be other points as valid answers and examiner have given credit for any alternative approach or interpretation which they consider to be reasonable.

Solution 1 :-

a) The price of call option is given by
$C(S_t) = N(d_1)S_t – N(d_2)Ke^{-rt}$
where $d_1 = [\ln(S_t/K)+(r+\sigma^2/2)t]/\sigma\sqrt{t}$ and $d_2 = d_1 - \sigma\sqrt{t}$
Thus at time 0:
$d_1 = [0+(.05+.4^2/2)(30/365)]/ [0.4\sqrt{(30/365)}] = 0.0932$
$d_2 = -0.0215$
$N(d_1) = 0.5371$
$N(d_2) = 0.4914$
$C(S_0) = 0.5371 \times 100 – 0.4914 \times 100 \times \exp[-0.05(30/365)] = 4.771$

b) The delta is given by $\Delta_0 = N(d_1) = 0.5371$

c) The trader should buy $\Delta_0 S_0 = 0.5371 \times 100 = 53.71$ worth of stock S on day 0

d) The trader would need to borrow the difference between the cost of buying the stock and the price of the call option. Hence his cost of borrowing will be
$[\exp(0.05/365)-1] \times (53.71-4.771) = 0.0067$, which is negligible

e) At day 1, we have $t = 29/365$ and $S = S_1 = 105$
Using previously stated formula, we get $C(S_1) = 7.7931$

Thus, the traders profit =
Loss due to increase in call price + Gain due to increase in stock price - Cost of borrowing
= -(7.7913-4.771) + 0.5371*5-0.0067 = -0.3415 ~ -0.34

Hence the trader will make a loss of 0.34

[Total 9 Marks]

Solution 2 :-

i. Lagrangian function $W = V – \lambda(E - E_p) – \mu(x_A + x_B – 1)$

$\Rightarrow W = (x_A^2V_A + x_B^2V_B + 2x_Ax_BC_{AB}) - \lambda(x_AE_A + x_BE_B - E_P) – \mu(x_A + x_B – 1)$
$\Rightarrow W = (x_A^2\sigma_A^2 + x_B^2\sigma_B^2 + 2x_Ax_B\rho_{AB}\,\sigma_A\sigma_B) - \lambda(x_AE_A + x_BE_B - E_P) – \mu(x_A + x_B – 1)$
$\Rightarrow W = (4x_A^2 + 9x_B^2 + 2x_Ax_B \times 0.25 \times 2 \times 3) - \lambda(2x_A + 5x_B - E_P) – \mu(x_A + x_B – 1)$
$\Rightarrow W = (4x_A^2 + 9x_B^2 + 3x_Ax_B) - \lambda(2x_A + 5x_B - E_P) – \mu(x_A + x_B – 1)$

ii. The first order conditions are obtained by setting the partial derivatives of W wrt x_A, x_B, λ and μ equal to zero.

$$\frac{\partial W}{\partial x_A} = 8x_A + 3x_B - 2\lambda - \mu = 0;$$

$$\frac{\partial W}{\partial x_B} = 18x_B + 3x_A - 5\lambda - \mu = 0;$$

$\frac{\partial w}{\partial \lambda}$ = -($2x_A + 5x_B - E_P$) = 0;

$\frac{\partial w}{\partial \mu}$ = -($x_A + x_B - 1$) = 0

Solving the last two equations we get $x_B = 1 - x_A$ and $2x_A + 5(1 - x_A) = E_P$
⇒ $x_A = (5 - E_P)/3$ and $x_B = (E_P - 2)/3$

⇒ $V = 4x_A^2 + 9x_B^2 + 3x_Ax_B = 4 \times \frac{(5-E_P)^2}{9} + 9 \times \frac{(E_P-2)^2}{9} + 3 \times \frac{(5-E_P)}{3} \times \frac{(E_P-2)}{3}$
⇒ $V = \frac{1}{9}(10E_P^2 - 55E_P + 106)$

To minimise variance, differentiate V wrt E_P and equate to zero
$\frac{dV}{dE_P} = 20E_P - 55 = 0$
⇒ E_P = expected return at the point of global minimum variance = 2.75%

Hence, the proportions of the securities in the global minimum variance portfolio are $x_A = (5 - 2.75)/3 = 0.75$ and $x_B = (2.75 - 2)/3 = 0.25$

iii. $V = \frac{1}{9}(10E_P^2 - 55E_P + 106) = 3.375$
Standard deviation = 1.8371

The efficient frontier is the part of the minimum variance curve above the point of global minimum variance i.e. above (2.75%, 1.8371%) in expected return-standard deviation space.

[Total 9 Marks]

Solution 3 :-

i) $B_t, t \geq 0$ is standard Brownian motion with $B_0 = 0$
Let $X_t = \frac{1}{2}(B_t^2 - t)$ We need to show $E[X_t|F_s] = X_s,\ s < t$
$E[X_t|F_s] = E\left[\frac{1}{2}(B_t^2 - t)\middle|F_s\right]$

For $s < t$, write $B_t = B_s + (B_t - B_s)$
⇒ $E[X_t|F_s] = E\left[\frac{1}{2}\{B_s + (B_t - B_s)\}^2 - \frac{1}{2}t\middle|F_s\right]$
$= E\left[\frac{1}{2}\{B_s^2 + (B_t - B_s)^2 + 2B_s(B_t - B_s)\} - \frac{1}{2}t\middle| F_s\right]$

Since B_t is standard Brownian motion so conditioning on F_s, the value of B_s is known, time t is fixed and $(B_t - B_s)$ is independent of the history till time s.
⇒ $E[X_t|F_s] = \frac{1}{2}B_s^2 + B_s E(B_t - B_s) + \frac{1}{2}E[(B_t - B_s)^2] - \frac{1}{2}t$
Also, $B_t - B_s \sim N(0, t - s)$
⇒ $E(B_t - B_s) = 0$ and $V(B_t - B_s) = E[(B_t - B_s)^2] - 0^2 = t - s$

$\Rightarrow E[X_t|F_s] = \frac{1}{2}B_s^2 + 0 + \frac{1}{2}(t-s) - \frac{1}{2}t = \frac{1}{2}B_s^2 - \frac{1}{2}s = X_s$

ii) Let $X_t = \exp(23B_t - \frac{529}{2}t)$ We need to show $E[X_t|F_s] = X_s$, $s < t$

$E[X_t|F_s] = E\left[\exp\left(23B_t - \frac{529}{2}t\right)\middle| F_s\right]$

For $s < t$, write $B_t = B_s + (B_t - B_s)$

$\Rightarrow E[X_t|F_s] = E\left[\exp\left\{23(B_s + (B_t - B_s)) - \frac{529}{2}t\right\}\middle| F_s\right]$

Since B_t is standard Brownian motion so conditioning on F_s, the value of B_s is known and time t is fixed.

$\Rightarrow E[X_t|F_s] = \exp\left(23B_s - \frac{529}{2}t\right) E[\exp\{23(B_t - B_s)\}|F_s]$

$(B_t - B_s)$ is independent of the history till time s

$\Rightarrow E[X_t|F_s] = \exp\left(23B_s - \frac{529}{2}t\right) E[\exp\{23(B_t - B_s)\}]$

Also, $B_t - B_s \sim N(0, t-s) \Rightarrow E\left[e^{23(B_t - B_s)}\right] = e^{0\times 23 + \frac{1}{2}\times 23^2(t-s)}$

(using the formula of the moment generating function of a normal distribution)

$\Rightarrow E[X_t|F_s] = \exp\left(23B_s - \frac{529}{2}t\right) \times \exp\left[\frac{529}{2}(t-s)\right] = \exp\left(23B_s - \frac{529}{2}s\right) = X_s$

iii) $\{B_t, t \geq 0\}$ is standard Brownian motion

Then $\{B_1(t), t \geq 0\}$ defined by $B_1(t) = \frac{1}{\sqrt{c}}B_{ct}$ is also standard Brownian motion.

This is the scaling property of Brownian motion.

And $\{B_2(t), t \geq 0\}$ defined by $B_2(t) = tB_{1/t}$ is also standard Brownian motion.

This is the time inversion property of Brownian motion.

[Total 9 Marks]

Solution 4 :-

i)

Return	Asset A			Asset B		
	p	Σp	ΣΣp	p	Σp	ΣΣp
7%	0.2	0.2	0.2	0	0	0
6%	0.3	0.5	0.7	0.6	0.6	0.6
5%	0.5	1.0	1.7	0.4	1.0	1.6

From the above table, $\Sigma p_A(7\%) > \Sigma p_B(7\%)$ but $\Sigma p_A(6\%) < \Sigma p_B(6\%)$
So neither asset first-order dominates the other.
$\Sigma\Sigma p_A(7\%) > \Sigma\Sigma p_B(7\%)$, $\Sigma\Sigma p_A(6\%) > \Sigma\Sigma p_B(6\%)$ and $\Sigma\Sigma p_A(5\%) > \Sigma\Sigma p_B(5\%)$
So asset B second-order dominates asset A.

ii) The main advantage of using stochastic dominance to make investment decisions is that the investor's utility function need not be explicit and investment decisions can be made for a wide range of utility functions.

The main disadvantage is that it may be unable to choose between investments and generally compares two investments at a time, thus making it difficult to choose when considering a large number of available investments.

[Total 4 Marks]

Solution 5 :-

The risk-neutral probability for an upward movement in share price is

$$q = \frac{e^{0.015} - 0.97}{1.05 - 0.97} = 0.5639 \Rightarrow 1 - q = 0.4361$$

i) Share prices per the two-period binomial tree model:

733.163
698.25
665 677.303
645.05
625.699

Hence, European call option values:

33.163
18.422
10.234 0
0

0

Value of the two month European call = 10.234

ii) Same share prices as above. European put option values:

0

9.751

24.546 22.698

44.528

74.302

Value of the two month European put = 24.546

[Total 7 Marks]

Solution 6 :-

i. The probability density function for returns on the asset is

$$f(x) = \begin{cases} \frac{1}{6.5-(-1.5)} & ; -1.5 \leq x \leq 6.5 \\ 0 & ; otherwise \end{cases}$$

Expected shortfall below Ravi's benchmark of 3% = $\int_{-1.5}^{3} \frac{(3-x)}{8} dx$ = 1.2656%

Return on risk-free asset = 1%

Expected shortfall of the risk-free asset is max (3 - 1, 0) = 2%

Ravi would the risky asset to minimise his expected shortfall.

ii. Expected shortfall below Ravi's benchmark of 0% = $\int_{-1.5}^{0} \frac{(0-x)}{8} dx$ = 0.1406%

Expected shortfall of the risk-free asset is 0%

Ravi would choose the risk-free asset to minimise his expected shortfall.

Comments:

Expected shortfall from both assets decreases when benchmark decreases.

The risk-free asset is chosen when the required benchmark falls below the risk-free rate of return.

[Total 5 Marks]

Solution 7 :-

$S_t = S_0.\exp\,(0.05t + 0.4B_t)$, where S_t is the stock price after t years and B_t is standard Brownian motion.

i) $P[S_4 > 175|S_3 = 125] = P[(S_4/S_3) > 1.4]$
$= P[\exp(0.05 \times 4 + 0.4B_4 - 0.05 \times 3 + 0.4B_3) > 1.4]$
$= P[\exp\{0.05 + 0.4(B_4 - B_3)\} > 1.4]$
$= P[(B_4 - B_3) > (\ln(1.4) - 0.05)/0.4]$
$= P[(B_4 - B_3) > 0.7162]$

Now, $B_4 - B_3 \sim N(0,1)$ so the required probability is $1 - \Phi(0.7162) = 0.2369$

ii) $S_4/S_3 = \exp\{0.05 + 0.4(B_4 - B_3)\}$
$\Rightarrow S_4 = 125.\exp\{0.05 + 0.4(B_4 - B_3)\}$, given $S_3 = 125$

Since, $B_4 - B_3 \sim N(0,1) \Rightarrow S_4 \sim LN(0.05 + \ln(125)\,, 0.4^2) = LN(4.8783\,, 0.16)$

$\Rightarrow \text{var}[S_4|S_3 = 125] = \exp(2 \times 4.8783 + 0.16) \times (\exp(0.16) - 1)$
$= 3516.117 = 59.297^2$

[Total 6 Marks]

Solution 8 :-

European put option on a share with one month to expiry and an exercise price of Rs 18.00
Share price at expiry would either be Rs 22.50 or Rs 13.50
Risk-free force of interest is 1.5% per month

Option's worth at expiry is Rs 0 if share price then is Rs 22.50
Option's worth at expiry is Rs 4.50 if share price then is Rs 13.50

i) You can create a hedged position by purchasing n shares so that value of the portfolio at expiry is the same whether the share price goes up or down.
⇒ 22.50n + 0 = 13.50n + 4.50 ⇒ n = 0.5
So the portfolio's value at expiry is Rs 11.25 in either case.

Let the price of the put be p
If the share is currently priced at Rs 20.44, then p + (20.44 x 0.5) = 11.25$e^{-0.015}$
⇒ p = 0.8625

ii) Let q denote the risk-neutral probability of an upward move of the share price
If the current share price is Rs 20.46, then 22.50 q + 13.50 (1 – q) = 20.46 $e^{0.015}$ ⇒ q = 0.8077

So the price of the put option is given by $e^{-0.015}$[0 x q + 4.50 x (1 – q)] = 0.8525

iii) The option's delta = $\Delta = \frac{\partial f}{\partial s}$ = (0.8625 - 0.8525) / (20.44 - 20.46) = -0.5

This was expected as we have shown in part i that creating a delta hedged position required purchasing 0.5 shares.

[Total 7 Marks]

Solution 9 :-

i) A discrete time stochastic process X_0, X_1, X_2 … is said to be a martingale if $E[|X_n|] < \infty$ for all n, and $E[X_n | X_0, X_1, X_2 \dots, X_m] = X_m$ for all m < n.

In other words, the current value X_m of a martingale is the optimum estimator of its future value X_n.

ii) $E(X_{i+1}|X_i) = (X_i+1)p+(X_i-1)(1-p)$
$= X_i + (2p-1)$
$= X_i$ if and only if p = 0.5

a.

	-3	-2	-1	0	1	2	3
-3	1	0	0	0	0	0	0
-2	0.5	0	0.5	0	0	0	0
-1	0	0.5	0	0.5	0	0	0
0	0	0	0.5	0	0.5	0	0
1	0	0	0	0.5	0	0.5	0
2	0	0	0	0	0.5	0	0.5
3	0	0	0	0	0	0	1

[Total 7 Marks]

Solution 10 :-

$dX_t = -3X_t dt + dB_t$, where B_t is a standard Brownian motion

a) $m_t = E[X_t \mid X_0 = x]$
$\Rightarrow dm_t = E[dX_t \mid X_0 = x]$
$= E[-3X_t dt + dB_t \mid X_0 = x]$
$= -3E[X_t \mid X_0 = x]dt + 0$
$= -3m_t dt$

$\Rightarrow \frac{dm_t}{m_t} = -3dt \Rightarrow d(\log m_t) = -3dt$

Integrating over (0,t)
$\Rightarrow \log m_t - \log m_0 = -3(t-0)$

$\Rightarrow \log m_t - \log(E[X_0 \mid X_0 = x]) = -3t$
$\Rightarrow \log m_t - \log x = -3t$
$\Rightarrow m_t = xe^{-3t}$

b) Using Ito's lemma $dX_t^2 = 2X_t dX_t + \frac{1}{2} \times 2 \times (dX_t)^2 = 2X_t dX_t + (dX_t)^2$

$\Rightarrow dX_t^2 = 2X_t(-3X_t dt + dB_t) + (-3X_t dt + dB_t)^2$
$= -6X_t^2 dt + 2X_t dB_t + (0 + 0 + dB_t)^2$
$= -6X_t^2 dt + 2X_t dB_t + dt$

c) $s_t = E[X_t^2 \mid X_0 = x]$

$\Rightarrow ds_t = E[dX_t^2 \mid X_0 = x]$
$= E[(-6X_t^2 dt + 2X_t dB_t + dt) \mid X_0 = x]$ *(from part ii)*
$= -6E[X_t^2 \mid X_0 = x]dt + 0 + dt$
$= -6s_t dt + dt$
$= -(6s_t - 1)dt$

$\Rightarrow \frac{ds_t}{(6s_t - 1)} = -dt \Rightarrow \frac{6ds_t}{(6s_t - 1)} = -6dt \Rightarrow d[\log(6s_t - 1)] = -6dt$

Integrating over (0,t)

$\Rightarrow \log(6s_t - 1) - \log(6s_0 - 1) = -6(t - 0)$
$\Rightarrow \log(6s_t - 1) - \log(6x^2 - 1) = -6t$, because $s_0 = E[X_0^2 \mid X_0 = x] = x^2$
$\Rightarrow 6s_t - 1 = e^{-6t}(6x^2 - 1)$
$\Rightarrow 6s_t = 1 + 6x^2 e^{-6t} - e^{-6t}$
$\Rightarrow s_t = x^2 e^{-6t} + \frac{1}{6}(1 - e^{-6t})$

d) $v_t = Var[X_t \mid X_0 = x]$

$= E[X_t^2 \mid X_0 = x] - E^2[X_t \mid X_0 = x]$
$= s_t - (m_t)^2$
$= x^2 e^{-6t} + \frac{1}{6}(1 - e^{-6t}) - x^2 e^{-6t}$
$= \frac{1}{6}(1 - e^{-6t})$

[Total 7 Marks]

Solution 11 :-

Portfolio 1

E(r)Boom =
[(4,000 ÷ (4,000 + 6,000) × .18] + [6,000 ÷ (4,000 + 6,000) × .10] = .072 + .06 =0.132=13.2%

E(r)Normal =
[(4,000 ÷ (4,000 + 6,000) × .07] + [6,000 ÷ (4,000 + 6,000) × .08] = 0.028+0.048 =0.076

E(r)Portfolio = (.25 × .132) + (.75 × .076)=0.09
VarPortfolio = [.25 × (.132 − .09)2] + [.75 × (.076 − .09)2] = .000588

BetaPortfolio = [(4,000 ÷ (4,000 + 6,000) × .64] + [6,000 ÷ (4,000 + 6,000) × 1.04]
=0.256+0.888 = 0.88

Portfolio 2

E(r)Boom = (.30 × .12) + (.70 × .20) = .036 + .14 = .176
E(r)Normal = (.30 × .06) + (.70 × .04) = .018 + .028 = .046

E(r)Portfolio = (.40 × .176) + (.60 × .046) = .0704 + .0276 = .098
VarPortfolio = [.40 × (.176 − .098)2] + [.60 × (.046 − .098)2] = .0024336 + .0016224 = .004056

BetaPortfolio = (.30 × 0.36) + (.70 × 1.48) = 1.144

a) According to the CAPM, the expected return on a single risky stock (or a well diversified portfolio) is given by:
R = *Rf* + β(*RM*-*Rf*)
where
Rf is the risk free rate,
RM is the expected market return and
β is the covariance of the return of the risky stock (or well diversified portfolio) with the market, divided by the market variance.

For portfolio 1 we have, 0.09 = *Rf* + 0.88*(*RM*-*Rf*), so 0.09= 0.12R*f +0.88RM*
For portfolio 2 we have, 0.098 = *Rf* + 1.144*(*RM*-*Rf*), so 0.098= -0.144*Rf* + 1.144*RM*
Which is consistent with *Rf* = 0.063 and *RM* = 0.0936

b) The assumptions for CAPM to hold good are as follows:

- All investors focus on a single holding period, and they seek to maximize the expected utility of their terminal wealth by choosing among alternative portfolios on the basis of each portfolio's expected return and standard deviation
- Investors aim to maximize economic utilities
- All investors can borrow or lend an unlimited amount at a given risk-free rate of interest and there are no restrictions on short sales of any assets
- All investors can borrow or lend an unlimited amount at a given risk-free rate of interest and there are no restrictions on short sales of any assets
- All assets are perfectly divisible and perfectly liquid (that is, marketable at the going price)
- All investors are price takers (that is, all investors assume that their own buying and selling activity will not affect stock prices)
- The quantities of all assets are given and fixed
- There are no transaction costs or taxes

a) The investor can use Arbitrage Pricing theory that does not rely on the strong assumptions of CAPM. It is an equilibrium market model which requires that the returns on any stock is linearly related to a set of factor indices as below:

$R_{I =}\ a_i+b_{i,1}I_1 + b_{i,2}I_2+\ldots\ldots\ldots\ldots+b_{i,}I_L + c_i$

where

R_i is the return on security i

a_i and c_i are the constants and random parts respectively of the component of the return of security i

$I_1 \ldots I_L$ are the returns on a set of L indices

$B_{i,k}$ is the sensitivity of security i to index k

Also, $E(c_i)=0$, $E(c_{i,}c_j)=0$, for all i and j where $i \neq j$ and $E(c_i(I_J-E(I_J))=0$ for all stocks and indices

[Total 15 Marks]

<u>Solution 12</u> :-

a) The credit event is an event which will trigger the default of a bond and includes the following

- failure to pay either capital or coupon
- loss event
- bankruptcy
- rating downgrade of a bond by a rating agency

The outcome of a default may be that the contracted payment is

- rescheduled
- cancelled by the payment of an amount which is less than the default free value of the original contract

- Cancelled and replaced with freshly issued euity in the company
- Continued but at a reduced rate
- Totally wiped out

b) A reduced form model is a statistical model, which uses observed market statistics rather than specific data relating to the issuing corporate entity to model the movement of the credit rating of bonds issued by the corporate entity over time The J-L-T model utilizes such market statistics in the form of multiple state default likelihoods established from credit rating transition probabilities drawn from established rating agencies.

c) The JLT model assumes that the transition intensities between default states are deterministic. An adaptation could be to assume that the transition intensity between states is stochastic and dependent on a separate state variable process. By using the stochastic approach, the transition intensities could vary with various economic factors. For example, a rise in interest rates could increase default risk and so the variable process could include appropriate allowances for a change in interest rates.

d) Let Q(1) be the risk neutral probability that a corporate bond will default between time zero and 1 year. Assuming there is no recovery in the event of default the probability is Q(1) that the bond will be worth 0 at maturity and probability is 1-Q(1) that it will worth Rs. 100(principal amount). The expected value of the bond is therefore {Q(1)*0+[1-Q(1)]*100}*exp(-r_f) where r_f is the 1 year risk free zero rate. If the yield on the bond is r then 100.exp(-r)=100[1-Q(1]exp(-r_f) i.e. Q(1) = 1- exp(-r+ r_f). Hence for investment grade I, Q(1) = 1-exp(-.013) = 1.29%. For Junk grade Q(1) = 1.685%

e) The ratings transition matrix will be (the sum of transition probabilities from one state to the possible states is 1)

State	I	J	D
I	0.90	0.0871	0.0129
J	0.18315	0.8	0.01685
D	0	0	1

f) For recovery rate say R (R=40% and 10%), the risk neutral default probability = [1- exp(-r+ r_f)]/(1-R).Using this, the revised default probabilities will be for I = 0.03225 and for J =0.1685

State	I	J	D
I	0.90	0.06775	0.03225
J	0.0315	0.8	0.1685
D	0	0	1

[Total 15 Marks]