INSTITUTE OF ACTUARIES OF INDIA

EXAMINATIONS

May 2013

Subject : CT1 – Financial Mathematics

INDICATIVE SOLUTIONS

Sol. 1 :

(i)

$$F_t = \frac{-1}{Pt}\frac{\partial}{\partial t} P_t$$

$$= \frac{-1}{(100-1.5t^2)}(-3t)$$

$$F_3 = \frac{3*3}{100-1.5*9} = \frac{9}{86.5}$$

$$= 10.4046\%$$

(ii) (a)

$$(1 + f_{2,3})^3 = \frac{(1+ \quad y_5 \quad)^5}{(1+ \quad y_2 \quad)^2} = \frac{(1+(3.05+0.35*5)/100)^5}{(1+(3.05+0.35*2)/100)^2} = \frac{(1.048)^5}{(1.0375)^2}$$

$(1 + f_{2,3})^3 = 1.1744$

$(1 + f_{2,3}) = 1.05505$

$f_{2,3} = 5.51\%$

(ii) (b)

The n-year par yield represents the coupon per Rs.100 nominal that would be payable on a bond with term n years, which would give the bond a current price under the current term structure of Rs100 per Rs 100 nominal, assuming the bond is redeemed at par.

That is, if y_{cn} is the n-year par yield:

$$1 = y_{cn}\,(v_{y1} + v_{y2}^2 + v_{y3}^3 3 + \text{-------} + v_{yn}^n) + 1\, v_{yn}^n$$

(ii) (c)

Par yield is y_{c4} where

$$y_{c4}\left(v_{y1} + v_{y2}^2 + v_{y3}^3 + v_{y4}^4\right) + v_{y4}^4 = 1$$

$$y_{c4}(1.034^{-1} + 1.0375^{-2} + 1.041^{-3} + 1.0445^{-4}) + 1.0445^{-4} = 1$$

$$y_{c4}(0.9671 + 0.92902 + 0.88644 + 0.84017) + 0.84017 = 1$$

$$y_{c4} = \frac{0.1598}{3.6227} = 4.4111\% = 4.41\%$$

[6 Marks]

Sol. 2 :

Interest amount in t th installment = $P * (1 - v^{n-t+1})$

Loan repayment in t th installment = $P * v^{n-t+1}$

Where P is EMI and n is the total term of the loan

Interest amount in 16^{th} installment = $P * (1 - v^{10})$ where n = 25

Loan repayment in 21^{st} installment = $P * v^5$ where n = 25

$P * (1 - v^{10}) = P * v^5$

$1 - v^{10} = v^5$

$v^{10} + v^5 - 1 = 0$

$v^{10} + v^5 + ¼ = 1 + ¼$

$(v^5 + ½)^2 = (\sqrt{5}/2)^2$

$v^5 = (\sqrt{5}/2) - ½$ or $v^5 = -(\sqrt{5}/2) - ½$

Or $v^5 = \frac{-b+\sqrt{b^2-4ac}}{2a}$ or $v^5 = \frac{-b-\sqrt{b^2-4ac}}{2a}$ = $v^5 = (\sqrt{5}/2) - ½$ or $v^5 = -(\sqrt{5}/2) - ½$

As interest rate should be positive

v = 0.90824 **[5 Marks]**

i = 10.102588% = 10.1026%

Sol. 3 :

275 * $s_{\overline{n}|}^{(4)}$ * $(1+i)^{4n}$ + 450 * $s_{\overline{4n}|}^{(4)}$ = 24000

275 * $\frac{(1+i)^{n}-1}{i^{(4)}}$) * $(1+i)^{4n}$ + 450 * ($\frac{(1+i)^{4n}-1}{i^{(4)}}$) = 24000

275 * ($\frac{(1.5)-1}{i^{(4)}}$) * $(1.5)^4$ + 450 * ($\frac{(1.5)^4-1}{i^{(4)}}$) = 24000

137.5 * $(1.5)^4$ + 450 *($(1.5)^4$ – 1) = 24000 *$i^{(4)}$
696.094 + 1828.125 = 24000 * $i^{(4)}$
$i^{(4)}$ = 2524.219/24000 = 10.5176%
i.e. i = 10.9397%

[4 Marks]

Sol. 4 :

(i) <u>Fixed interest government borrowing</u>

a. Fixed interest government bonds
- Issued at a price or by tender
- Investors receive coupons (usually half-yearly) plus redemption payment
- Redemptions are usually at par
- Some redemption dates are variable and some stocks are undated
- Very secure, liquid and marketable (in developed countries)
- Low expected return
- Low dealing costs
- Real returns are uncertain.

b. Government bills
- Short-term usually less than one year
- Issued at a discount, redeemed at par
- No coupons
- Very marketable and secure
- Yield quoted as simple rate of discount

(ii) Cash flows to borrower fixed interest repayment loan :

Positive cash flow as initial lump sum amount while availing loan

Periodic outgo of fixed repayment amount during the term of the loan

Both the amount and the timing of repayments are known in advance

(iii)

Positive Inflation – The money rate of interest will be higher than the real rate of interest
Negative Inflation – The money rate of interest will be lower than the real rate of interest.

(iv)

Present value of the dividends = 0.30 * $v^{1/2}$ @ 7% + 0.30 v @ 7.5%
= 0.30 * (0.9667 + 0.9302)
= 0.56907
Forward Price = (12 – 0.56907) * (1.075) = ₹ 12.29

(v)

Test for capital gain :
$i^{(2)} = 2*((1+i)^{(1/2)} - 1) = (1.08)^{(1/2)} - 1 = 7.8461\%$

$g(1-t) = D/R(1-t) = (6/130)(1-0.2) = .036963 < i^{(2)}$
Hence investor has capital gain
$P = 6 * (1 - 0.20)\, a^{(2)}_{\overline{11}|} + 130 * v^{11} - (130 - P) * 0.30 * v^{11}$ @ 8% p.a.

$V^{11} = 0.42888$

$a^{(2)}_{\overline{11}|} = \frac{1-v^{11}}{i^{(2)}} = \frac{1-v^{11}}{i^{(2)}} = \frac{1-0.4289}{0.078461} = 7.278993$

P = 6 * 0.8 * 7.27899 + 130 * 0.42888 – (130 – P) * 0.30 * 0.42888
P = 34.93918 + 55.75477 – 16.72632 + 0.128664 P
P = 73.96763/(1 – 0.128664) = 84.8899

[16 Marks]

Sol. 5 :

(i) $A(0,10) = \text{Exp} \int_0^{10} 0.05 = Exp\,(0.5) = 1.64872$

$$A(10,20) = \text{Exp} \int_{10}^{20} .006t\,dt = Exp\,(1.2 - 0.3) = Exp\,(0.9) = 2.4596$$

$$A(20,25) = Exp \int_{20}^{25} 0.003t + 0.0002t^2\,dt$$
$$= \exp(0.9375 + 1.04166 - 0.6 - 0.5333 = \exp(0.84586) = 2.32$$

Required PV = 100 / (A(0,10) A(10, 20) A(20, 25))

= 100/ (1.64872 * 2.45960 *2.32998 = 1/ 9.4485
= 10.584

(ii) $A(19,20) = EXP\ (\int_{19}^{20} .006\ t\ dt\ ) = \exp(\ 1.2 - 1.083) = 1.12412$

i = 12.412%

(iii) $V(t) = Exp - \ (\ \int_0^t 0.05\ ds = \exp(\ -0.05t)$

$P(t) = \ exp\ (-0.03t)$

We Require

$\int_0^5 exp\ (\ -0.05t) * exp\ (\ -0.03t)dt\ = \ \int_0^5 exp\ (-0.08t)dt = \ \ 4.121$

[7 Marks]

Sol. 6 :

(i) Let the liability consist of a lump sum payment of X at time t and lump sum payment of 2.5 X at time t+3

Present Value of liabilities = X v^t + 2.5 X $v^{(t+3)}$ = 7107.77

$Xv^t = \frac{7107.77}{(1+2.5\ \ v^3)}$ - Equation (1)

Discounted Mean Term = $\frac{\mathrm{X}\ t\ v^t\ \ + 2.5\ \mathrm{X}\ (t+3)v^{t+3}}{\mathrm{X}\ v^t\ + 2.5\ \mathrm{X}\ v^{t+3}}$

DMT = $\frac{\mathrm{X}\ v^t\ \ (\ \mathrm{t} + 2.5\ (t+3)v^3)}{7107.77}$ = 5

Substituting from Equation (1) $\frac{\mathrm{X}\ v^t}{7107.77} = \frac{1}{(1+2.5\ v^3)}$

$$5 = \frac{\left(t + 2.5\,(t+3)v^3\right)}{1 + 2.5\,v^3} = \frac{(t + 2.5*\;(t+3) * 0.7938)}{1\;+ 2.5 *0.7938}$$

$$= \frac{(t + 1.9845\;t + 5.9535)}{1 + 1.9845}$$

Time t = 3.0052 ; In complete years t = 3

Payments are made at the end of 3rd year and 6th year from now.

$$X = \frac{7107.77}{(1+2.5 *0.7938)*0.7938}$$

First Payment X = 3000 and second payment 2.5X = 7500.

(ii) The conditions for Redington's immunisation may be summarized as follows:
At the initial rate of interest,

1. $V_A(i_0) = V_L(i_0)$ – that is, the present value of the assets is equal to the present value of the liabilities.

2. The volatility (or DMT) of the assets is equal to the Volatility (or DMT) of liabilities.

3. The convexity of the assets is greater than the convexity of the liabilities.

(iii) P.V. of the asset = $10443.65 / (1.08)^5$

= 7107.77

which is the same as the present value of the liabilities. Hence the first condition of immunization is satisfied.

Duration of the zero coupon bond is the term itself, which is 5 years. Hence the durations of the Liabilities and Assets are equal. Thus the second condition of immunization is also satisfied.

The convexity of the assets will be less than the convexity of the liabilities because the asset cash flow falls between the liability cash flows. Hence the third condition of Redington's theory is not met. Overall the portfolio is not immunized against small changes in interest rates.

[13 Marks]

Sol.7 :

(i) The cash flow for increasing annuity is as follows:

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Continuous payments    1        2      3                    n-1       n
                      ←=→     ←=→    ←→                               ←=→
                     |---------|---------|--------|------------------------|---------|
          time       0         1         2        3                       n-1        n
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Breaking it into each individual year

$$(I\,\bar{a})_{\overline{n|}} = \int_0^1 v^t\,dt + \int_1^2 2v^t\,dt + \ldots\ldots\ldots + \int_{n-1}^n nv^t\,dt$$

The value at time t of an annuity payable continuously between time t and time t+1, where the rate of payment per unit time is constant and equal to 1, is denoted by $\bar{a}_{\overline{1|}}$, where

$\bar{a}_{\overline{1|}} = \int_0^1 v^t\,dt = \int_0^1 e^{-\delta t}\,dt = [1 - e^{-\delta}]/\delta$

= [1-v]/δ

$$(I\,\bar{a})_{\overline{n|}} = \bar{a}_{\overline{1|}} + 2\,v\bar{a}_{\overline{1|}} + \ldots\ldots\ldots + n\,v^{n-1}\,\bar{a}_{\overline{1|}} \quad \text{-------(i)}$$

Multiplying both sides by v

$$v\,(I\,\bar{a})_{\overline{n|}} = v\,\bar{a}_{\overline{1|}} + 2\,v^2\bar{a}_{\overline{1|}} + \ldots\ldots\ldots + n\,v^n\,\bar{a}_{\overline{1|}} \quad \text{-------(ii)}$$

Subtracting (ii) from (i)

$$(1-v)\,(I\,\bar{a})_{\overline{n|}} = \bar{a}_{\overline{1|}}\,(1 + v^1 + v^2 + \ldots\ldots\ldots + v^{n-1} - nv^n)$$

$$(1-v)\,(I\,\bar{a})_{\overline{n|}} = \bar{a}_{\overline{1|}}\,(\ddot{a}_{\overline{n|}} - nv^n)$$

$$(1-v)\,(I\,\bar{a})_{\overline{n|}} = \frac{1-v}{\delta}\,(\ddot{a}_{\overline{n|}} - nv^n)$$

$$(I\,\bar{a})_{\overline{n|}} = \frac{(\ddot{a}_{\overline{n|}} - nv^n)}{\delta}$$

(ii) (a)

PV of all the withdrawals at the end of 48th month = PV of 12 quarterly withdrawals

$= (510-10)v + (510-20)v^2 + (510-30)v^3 + \dots\dots\dots + (510-120)v^{12}$ @ i'

$= 510\, a_{\overline{12|}} - 10\, (Ia)_{\overline{12|}}$ @ i'

Where i' = Quarterly effective rate = $((1 + 0.04)^{(1/4)} - 1 = 0.98534\%$

$$a_{\overline{12|}} = \frac{1-v^{12}}{i} = \frac{1-0.990243^{12}}{0.0098534} = 11.2652$$

$$\ddot{a}_{\overline{12|}} = \frac{1-v^{12}}{d} = \frac{1-0.990243^{12}}{0.009757} = 11.3765$$

$$(Ia)_{\overline{12|}} = \frac{(\ddot{a}_{\overline{12|}} - 12v^{12})}{i} = \frac{(\ddot{a}_{\overline{12|}} - 12v^{12})}{i} = \frac{(11.3765 - 12 * 0.990243^{12})}{0.0098534}$$

= 71.9051

PV at 48th month = 510 * 11.2652 – 10 * 71.9051 = 5026.201

(b)

Amount available in the deposit at the end of 72nd month
= Accumulation of initial deposit – Accumulation of 8 quarterly withdrawals

Accum of 8 quarterly withdrawals = $(510\, a_{\overline{8|}} - 10\, (Ia)_{\overline{8|}}$ @ i')$(1.04)^2$
Where i' = $((1 + 0.04)^{(1/4)} - 1 = 0.0098534$

$$a_{\overline{8|}} = \frac{1-v^{8}}{i} = \frac{1-0.990243^{8}}{0.0098534} = 7.6564$$

$$\ddot{a}_{\overline{8|}} = \frac{1-v^{8}}{d} = \frac{1-0.990243^{8}}{0.009757} = 7.7321$$

$$(Ia)_{\overline{8|}} = \frac{(\ddot{a}_{\overline{8|}} - 8v^{8})}{i} = \frac{(\ddot{a}_{\overline{8|}} - 8v^{8})}{i} = \frac{(7.7321 - 8 * 0.990243^{8})}{0.0098534} = 34.0628$$

Accum of 8 quarterly withdrawals = $(510 * 7.6564 - 10 * 34.0628)(1.04)^2$
$= 3564.136(1.04)^2 = 3854.9695$

Accumulation of initial deposit = $6000(1.04)^6 = 7591.9141$

Amount available in the deposit at the end of 72nd month
=7591.9141 – 3854.9695 = 3736.94

Sol.8 :

(i) (a) The *discounted payback period* for an investment project is the earliest time after the start of the project when the accumulated value (or Present Value) of the past cashflows (positive and negative), calculated using the borrowing rate, becomes greater or equal to zero. i.e., *Acc. profit at discounted payback period* ≥ 0 (*@ borrowing rate*)

(b) The payback period is the earliest point of time since start of the project at which total cash inflow is greater than total cash outflow.

(ii)

- Discounted payback period or payback period is an inappropriate decision criterion because it does not tell us anything about the overall profitability of the project.

- Discounted payback period or payback period does not take account either of interest on borrowings or on cash flows received after the payback period.

- The discounted payback period simply shows when a project became profitable in present value terms, not how profitable it is. However, this information can be useful in the decision making process.

- The payback period ignores interest altogether and is therefore clearly an inferior criterion. It is also possible for the discounted payback period and the payback period to be before the end of the project but the NPV can be negative.

- There may not be one unique time when the balance in the investor's account changes from negative to positive. However, the NPV can always be calculated. The PP can give misleading results, as it does not take into account the time value of money.

(iii)

PV of outflow = 180,000 $\ddot{a}_{\overline{3}|}$ @ 7%

$$\ddot{a}_{\overline{3}|} = \frac{1-v^3}{d} = \frac{1-0.9346^3}{0.06452}$$

PV of outflow = 180,000 × 2.80802

= 505,443.2701

PV of inflow = 25,000 * $\int_0^{25} (1.06)^t * (1.07)^{-t} dt$

This is equivalent to present value of annuity with v^t = (1.06/ 1.07)^t

= 0.990654^t i.e., i = 0.943396%

= 25,000 $\bar{a}_{25|}$ @ i = 0.943396%

= 25,000* (1- 0.990654^{25}) / δ   δ = 0.00938974

= 25,000* 22.28244

= 557,061.06

NPV = PV of inflow – PV of outflow = 557,061.0636 – 505,443.2701

= ₹ 51617.79

(iv)

Need to find t such that

505,443.2701 = 25,000 * (1 – 0.990654^t) / 0.00938974

0.189839 = 1 - 0.990654^t

Ln (0.189839 – 1) = t * ln (-99.0654)

t = 22.42 years

[15 Marks]

Sol.9:

(i)

Option A

$(1+ i_t) \sim \text{Lognormal}(\mu,\sigma^2)$

$\ln(1+ i_t) \sim N(\mu,\sigma^2)$
$\ln(1+ i_t)^5 = \ln(1+ i_t) + \ln(1+ i_t) + \ldots + \ln(1+ i_t) \sim N(5\mu,5\sigma^2)$
since i_t ' s are independent
$(1+ i_t)^5 \sim \text{Lognormal}(5\mu,5\sigma^2)$

$E(1 + i_t) = \exp(\mu + \sigma^2/2) = 1.075$

$Var(1 + i_t) = \exp(2\mu + \sigma^2) * (\exp(\sigma^2) - 1) = 0.03^2$

$(\exp(\sigma^2) - 1) = \frac{0.03^2}{1.075^2} = 0.000779$

$\sigma^2 = 0.0007787$

$\exp(\mu + 0.0007787/2) = 1.075$

$\mu = 0.071931$

$5\ \mu = 0.3597$

$5\ \sigma^2 = 0.003894$

$E(S_{10}) = (1.05)^5 * \exp(0.3597 + 0.003894/2)$

$E(S_{10}) = 1.83235$

$Var(S_{10}) = (1.05)^{10} * \exp(2 * 0.3597 + 0.003894) * (\exp(0.003894) - 1) = 0.0130996$

Therefore Standard Deviation= $0.0130996^{(1/2)} = 0.114453$

Option B

Expected Value at end of 7 years = $1.05^7 = 1.4071$

Expected Value at end of 10 years = $E[S_{10}]$

$= 1.4071 * (0.30 * 1.04^3 + 0.60 * 1.05^3 + 0.10 * 1.06^3)$

$= 1.6198$

$E[S_{10}^2] = 1.4071^2 * (0.30 * 1.04^6 + 0.60 * 1.05^6 + 0.10 * 1.06^6)$

$= 2.6244$

$Var(S_{10}) = E(S_{10}^2) - (E[S_{10}])^2 = 2.6244 - 1.6198^2 = 0.000648$

Therefore Standard Deviation $0.000648^{(1/2)} = 0.02546$

(ii)

Option A

Let S' be the random variable representing the Accumulated value from period 5 to 10

we require $P[S' < (1.65/1.05^5)] = P[S' < 1.2928]$

We require $P[S' < 1.2928]$ where $Ln(S') \approx N(0.3597, 0.003894)$

$P[N(0,1) < \frac{\ln(1.2928) - 0.3597}{0.003894^{(1/2)}}] = P[N(0,1) < -1.6488] \approx 4.96\%$

Option B

S_{10} With Probability 30% $= 1.05^7 * 1.04^3 = 1.5828$

S_{10} With Probability 60% $= 1.05^7 * 1.05^3 = 1.6289$

S_{10} With Probability 10% $= 1.05^7 * 1.06^3 = 1.6759$

$P[S_{10} < 165/100] = 90\%$ (= sum of first two = 30% + 60%)

[20 Marks]

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